Counter-Examples to the Concentration-Cancellation Property

Christophe Cheverry, Olivier Guès

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Christophe CHEVERRY [1], Olivier GUÈS [2]

Abstract. We study the existence and the asymptotic behavior of large amplitude high-frequency oscillating waves subjected to the 2D Burger equation. This program is achieved by developing tools related to supercritical WKB analysis. By selecting solutions which are divergence free, we show that incompressible or compressible 2D Euler equations are not *locally* closed for the weak L^2 topology.

1 Introduction.

This article is devoted to the study of the two dimensional incompressible Euler equation

$$(1.1) \qquad \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} + \nabla_x \mathbf{p} = 0\,, \qquad \mathrm{div}_x\, \mathbf{u} = 0$$

as well as to the study of the two dimensional Burger equation

$$(1.2) \qquad \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} + \mathbf{f} = 0\,.$$

The time, space and state variables are respectively

$$t \in \mathbb{R}\,, \qquad x = {}^t(x_1,x_2) \in \mathbb{R}^d\,, \qquad \mathbf{u} = {}^t(\mathbf{u}^1,\mathbf{u}^2) \in \mathbb{R}^d\,, \qquad d = 2\,.$$

The equations (1.1) and (1.2) are completed with some initial data having locally finite kinetic energy

$$(1.3) \qquad \mathbf{u}(0,x) = h(x) = {}^t\big(h^1(x), h^2(x)\big) \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)\,.$$

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1.1 The main result.

The analysis of the Cauchy problems (1.1)-(1.3) and (1.2)-(1.3) depends strongly on the regularity assumptions imposed on the function h. For instance, consider (1.1)-(1.3). When

$$h \in E_s \, := \, \big\{ \, v \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2) \, ; \, \operatorname{curl} v \, := \, \partial_1 v^2 - \partial_2 v^1 \in (L^1 \cap L^\infty)(\mathbb{R}^2;\mathbb{R}) \, \big\}$$

the solution $\mathbf{u}$ of (1.1)-(1.3) is global in time and is unique [2]-[4]-[24]. Now, since the equation (1.1) can be put in the conservative form

$$\partial_t \mathbf{u} + \operatorname{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \mathbf{p} \, = \, 0 \, , \qquad \operatorname{div}_x \mathbf{u} = 0 \, , \qquad \mathbf{u} \otimes \mathbf{u} := (\mathbf{u}^j \, \mathbf{u}^i)_{1 \leq i,j \leq 2} \, ,$$

one is tempted to work in a more general functional framework. This means to enter the field of *weak* solutions to (1.1). In the case of vortex-sheet initial data with vorticity of distinguished sign

$$h \in E_w \, := \, \big\{ \, v \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2) \, ; \; 0 \, \leq \, \operatorname{curl} v \in \mathcal{M}(\mathbb{R}^2;\mathbb{R}) \, \big\} \, , \tag{1.4}$$

existence results hold whereas the question of uniqueness is still open. The first proof is due to J.-M. Delort [10]. Then, further informations have been obtained. For background and expository accounts on this subject, the reader may consult [2], [22] and the related references.

The usual way (see for instance [2], [10] and [22]) to derive existence results of weak solutions to (1.1) is based on two steps. First, construct *approximate-solution* sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in]0,1]}$ of (1.1), either by smoothing the initial data h or by adding to (1.1) a small viscosity (to get Navier-Stokes equations). Secondly, exhibit a property of *concentration-cancellation* which means that the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$ does not converge (as ε goes to 0) strongly in L^2_{loc} yet all the extracted L^2_{loc} weak limits still satisfy (1.1).

This approach is well presented and clearly explained in the recent book of L. Bertozzi and A. Majda [2] (read especially the surveys given in chapters 10, 11 and 12). To achieve the second step, the difficulty is to identify the limit of the nonlinear terms contained in the expression $\operatorname{div}_x (\mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon)$. At this level, the articles [10] and [22] exploit in a crucial way the informations which can be deduced from the regularity assumption (1.4).

Now, up to the present, when $d = 2$, the weak limits of approximate solutions of (1.1) have always been observed to be solutions of (1.1). Thus, one can ask if the property of concentration-cancellation does generalize to the space L^2 of energy estimates. This question is explicitly raised in [2]-p. 479. It makes sense also in the less restrictive framework L^2_{loc}.

However, such a program comes again a deep objection because:

Theorem 1.1. *There is a bounded open domain $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ and a family of functions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ such that*

i) $\mathbf{u}^\varepsilon \in C^1(\Omega)\,, \qquad \sup \big\{ \parallel \mathbf{u}^\varepsilon \parallel_{L^\infty(\Omega)} ; \ \varepsilon \in]0,1] \big\} < \infty\,,$

ii) $\mathbf{u}^\varepsilon$ *is a solution of (1.1) on* Ω*,*

iii) $\mathbf{u}^\varepsilon$ *converges weakly (as* ε *goes to zero) to* $\mathbf{u}^0 \in C^1(\Omega)$*.*

But $\mathbf{u}^0$ *is not a solution of (1.1).*

When the space dimension is three ($d = 3$), the corresponding result can easily be proved, see [11]-p. 674 or [2]-p. 478. It suffices to look at *simple waves* which undergo rapid variations with respect to a *linear* phase $k \cdot x$ where $k \in \mathbb{R}^3$. Note $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ a fast variable. Build the oscillation

$$\mathbf{u}^\varepsilon(t,x) \,=\, \mathbf{H}(t,x,k \cdot x/\varepsilon)\,, \qquad \mathbf{H}(t,x,\theta) \in C^\infty(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{T}; \mathbb{R}^3)\,. \tag{1.5}$$

The profile $\mathbf{H}$ can be adjusted so that $\mathbf{u}^\varepsilon$ is a solution of (1.1) and (1.2) with respectively $\mathbf{p}$ constant and $\mathbf{f} \equiv 0$. But the associated weak limit $\mathbf{u}^0$ is not a solution to (1.1). Observe that

$$\mathbf{u}^0(t,x) \,=\, \bar{\mathbf{H}}(t,x) \,:=\, \textstyle\int_0^1 \mathbf{H}(t,x,\theta)\; d\theta \,=\, \mathbf{H}(t,x,\theta) - \mathbf{H}^*(t,x,\theta)\,.$$

When $d = 2$, such a basic procedure does not apply. Of course, there are solutions to (1.1) of the form (1.5) with $x \in \mathbb{R}^2$ and $k \in \mathbb{R}^2$. However, all known examples of such solutions produce weak limits which satisfy (1.1). Thus, to push further the investigations, it is necessary to take into account more general structures including *nonlinear* phases and *perturbations* of simple waves. On this way, new difficulties appear.

In fact, the study of solution sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in]0,1]}$ to (1.1) can reveal very complex phenomena. The asymptotic behavior of $\mathbf{u}^\varepsilon$ when ε goes to zero can involve both concentrations and oscillations. Certainly, this is a current challenge to understand what happens in the limiting process. Our aim here is precisely to bring informations of this type. This is achieved by following an original strategy which is presented below.

1.2 Compatibility conditions for initial data.

Section 2 is devoted to the Cauchy problem (1.2)-(1.3) which recently raised new interests [3]-[15]-[21]. The initial data h is defined on some open set $\omega \subset \mathbb{R}^2$. Moreover h and its derivatives up to the order one are bounded. We take $h \in C^1_b(\omega; \mathbb{R}^2)$. The source term $\mathbf{f}$ is globally defined on $\mathbb{R} \times \mathbb{R}^2$ and it is of class C^1, that is $\mathbf{f} \in C^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$.

To any parameter $\tau \in]0,1]$ is associated a set $\mathcal{B}^\tau(\omega)$ of admissible data $(h,\mathbf{f})$ and some non trivial domain $\Omega^\tau \subset \mathbb{R}^+ \times \mathbb{R}^2$ which contains $\{0\} \times \omega$. Lemma 2.2 says that if $(h,\mathbf{f}) \in \mathcal{B}^\tau(\omega)$ the local in time C^1 solution $\mathbf{u}$ of (1.2)-(1.3) can be extended to the whole domain Ω^τ.

Three special cases $\mathbf{f}_\iota \equiv \nabla_x \mathbf{p}_\iota$ $\big($with $\iota \in \{-1,0,1\}$ and $\mathbf{p}_\iota(t,x) := \frac{\iota}{2}\,|x|^2\big)$ for the choice of $\mathbf{f}$ are then distinguished. When $\mathbf{f} \equiv \mathbf{f}_\iota$, the conditions mentioned above concern only h and are noted in abbreviated form $h \in \mathcal{B}^\tau_\iota(\omega)$. In fact, they reduce to some control on the quantities h, $\operatorname{div}_x h$ and $\det D_x h$. This is the aim of Lemma 2.3 and of remarks 2.1.4 and 2.1.5.

Moreover, to each ι corresponds a nonlinear functional set $\mathcal{V}^\tau_\iota(\omega) \subset \mathcal{B}^\tau_\iota(\omega)$ which is defined through a *Monge-Ampère* equation and which is kept invariant (Lemma 2.4) under both flows issued from (1.1) and (1.2). Therefore, when restricted to $\mathcal{V}^\tau_\iota(\omega)$, all the discussion concerning (1.1) can be transfered at the level of (1.2). This argument will be used repeatedly.

An important point is that the conditions $h \in \mathcal{B}^\tau_\iota(\omega)$ and even $h \in \mathcal{V}^\tau_\iota(\omega)$ do not mean a uniform control on *all* derivatives of h. In particular, we can find large amplitude high-frequency oscillations $\{h^\varepsilon\}_{\varepsilon \in]0,1]}$ such that $h^\varepsilon \in \mathcal{B}^\tau_\iota(\omega)$ or $h^\varepsilon \in \mathcal{V}^\tau_\iota(\omega)$ for all $\varepsilon \in]0,1]$. This gives the idea to perform a nonlinear geometric *under constraint*, the nonlinear constraint being given by $h^\varepsilon \in \mathcal{B}^\tau_\iota(\omega)$ or $h^\varepsilon \in \mathcal{V}^\tau_\iota(\omega)$. Section 2 is devoted to the existence of such oscillating initial data.

- In subsection 2.2, we identify (Lemma 2.5) the necessary and sufficient conditions to impose on the phase ϕ and the profile H in order to have

$$(1.6) \qquad h^\varepsilon(x) := H\big(x, \phi(x)/\varepsilon\big) + O(\varepsilon) \in \mathcal{B}^\tau_\iota(\omega), \qquad \forall\, \varepsilon \in]0,1].$$

As usual, the profile $H \in C^\infty(\omega \times \mathbb{T}; \mathbb{R}^2)$ must be well polarized

$$(1.7) \qquad \nabla_x \phi(x) \cdot H^*(x,\theta) = 0, \qquad \forall\, (x,\theta) \in \omega \times \mathbb{T}.$$

Moreover, the phase ϕ must satisfy the geometric condition

$$(1.8) \qquad \exists\, f; \qquad \partial_1 \phi(x) = f\big(\phi(x)\big)\, \partial_2 \phi(x), \qquad \forall\, x \in \omega$$

and the mean part $\bar H = {}^t(\bar H^1, \bar H^2)$ of H must be subjected to

$$(1.9) \qquad \exists\, g; \qquad f\big(\phi(x)\big)\, \bar H^1(x) + \bar H^2(x) = g\big(\phi(x)\big), \qquad \forall\, x \in \omega.$$

- In subsection 2.3, we extract (Lemma 2.7) supplementary constraints (on ϕ and H) which allow to get the more restrictive condition

$$(1.10) \qquad h^\varepsilon(x) := H\big(x, \phi(x)/\varepsilon\big) + O(\varepsilon) \in \mathcal{V}^\tau_0(\omega), \qquad \forall\, \varepsilon \in]0,1].$$

Couples (H,ϕ) adjusted so that (1.6) or (1.10) are verified are called *compatible* with respectively (1.2) or (1.1). The corresponding families $\{h^\varepsilon\}_\varepsilon$ are said *well prepared* for respectively the Burger equation or the Euler equation. Corresponding to such well prepared initial data h^ε, there are solutions of (1.2) or (1.1) which exist on the whole domain Ω^τ.

1.3 Existence of simple waves.

The chapter 3 is devoted to the existence and the construction of simple waves which are solutions of (1.2) or (1.1). What we call a *simple wave* is an expression of the form

$$\tilde{\mathbf{u}}^\varepsilon(t,x) \,=\, \mathbf{H}\big(t,x,\Phi(t,x)/\varepsilon\big)\,, \qquad \varepsilon\in\,]0,1] \tag{1.11}$$

where $\mathbf{H}$ and Φ do not depend on the parameter $\varepsilon\in\,]0,1]$.

- In the subsection 3.1, we seek *all* solutions to (1.2) which are as in (1.11). We show that the functions $\mathbf{H}$ and Φ must satisfy the following system

$$\left\{ \begin{array}{ll} \partial_t\mathbf{H} \,+\, (\mathbf{H}\cdot\nabla_x)\mathbf{H} \,+\, \mathbf{f} \,=\, 0\,, & \qquad \mathbf{H}(0,\cdot)=H\,, \\ \partial_t\Phi \,+\, (\bar{\mathbf{H}}\cdot\nabla_x)\Phi \,=\, 0\,, & \qquad \Phi(0,\cdot)=\phi\,, \\ \nabla_x\Phi(t,x)\cdot\mathbf{H}^*(t,x,\theta) \,=\, 0\,. & \end{array}\right. \tag{1.12}$$

Of course, this system (1.12) is also sufficient for the function $\tilde{\mathbf{u}}^\varepsilon$ in (1.11) to be a solution of (1.2). The first two equations in (1.12) allow to determine uniquely the couple $(\mathbf{H},\Phi)$. Then, it remains the third condition which is not sure to be satisfied. In fact, for general initial data satisfying $\nabla_x\phi\cdot H^*=0$, the problem (1.12) has no solution.

Nevertheless, the system (1.12) can be solved when (H,ϕ) is compatible with (1.2) and when $\mathbf{f}$ is subjected to conditions which will be made explicit (Lemma 3.1). In that case, the profile $\mathbf{H}$ can be written (Lemma 3.2)

$$\mathbf{H}(t,x,\theta) \,=\, H_\perp(\Phi) \,+\, \mathbf{r}(t,x,\theta)\ {}^t\big(-1,f(\Phi)\big)\,.$$

In this formula, the function $\mathbf{r}\in C^\infty(\Omega^\tau\times\mathbb{T};\mathbb{R})$ is determined through the scalar conservation law (3.10) and

$$H_\perp(z) \,:=\, \big[1+f(z)^2\big]^{-1}\, g(z)\ {}^t\big(f(z),1\big)\,.$$

Note that the relations (1.8) and (1.9) are preserved during the evolution

$$\partial_1\Phi(t,x) \,=\, f\big(\Phi(t,x)\big)\,\partial_2\Phi(t,x)\,, \qquad \forall\,(t,x)\in\Omega^\tau\,, \tag{1.13}$$

$$f\big(\Phi(t,x)\big)\,\bar{\mathbf{H}}^1(t,x) \,+\, \bar{\mathbf{H}}^2(t,x) \,=\, g\big(\Phi(t,x)\big)\,, \qquad \forall\,(t,x)\in\Omega^\tau\,. \tag{1.14}$$

In particular, this implies that $\Phi(t,x)$ can be determined without appealing to the construction of $\mathbf{H}$, just by solving

$$\partial_t \Phi \,+\, g(\Phi)\,\partial_2 \Phi \,=\, 0\,, \qquad \Phi(0,\cdot) = \phi\,. \tag{1.15}$$

- In subsection 3.2, we consider solutions of (1.1) having the form (1.11). This time, the functions $\mathbf{H}$ and Φ must be subjected to

$$\left\{ \begin{array}{ll} \partial_t \mathbf{H} \,+\, (\mathbf{H}\cdot\nabla_x)\mathbf{H} \,+\, \nabla_x \mathbf{p} \,=\, 0\,, & \operatorname{div}_x \mathbf{H} \,=\, 0\,, \\ \partial_t \Phi \,+\, (\bar{\mathbf{H}}\cdot\nabla_x)\Phi \,=\, 0\,, & \nabla_x \Phi \cdot \mathbf{H}^* \,=\, 0\,. \end{array} \right. \tag{1.16}$$

Again this is an over-determined system. The corresponding Cauchy problem associated with general initial data (H,ϕ) is not well posed. In Lemma 3.3, we detect the solutions of (1.12) which are also solutions of (1.16) with $\mathbf{p}\equiv\mathbf{p}_\iota$ for some $\iota\in\{-1,0,1\}$. In particular, the phase Φ must be constant on parallel hyperplanes which means that it is linear

$$\exists\,(a,b)\in\mathbb{R}^2\,; \qquad \Phi(t,x) \,=\, a\,t + b\,x_1 + x_2\,.$$

At this stage, weak limits of solutions to (1.1) of the form (1.11) are still solutions to (1.1). However, this will no more be true under the influence of suitable perturbations, as we will see in the last chapter 5.

1.4 The problem of stability.

Section 4 deals with the Burger equation (1.2). It investigates in this framework the *stability* of the family $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$. Select any solution $(\mathbf{H}^0,\Phi^0)$ of (1.12), as constructed in the chapter 3. Note

$$H(x,\theta) \,:=\, \mathbf{H}^0(0,x,\theta)\,, \qquad \phi(x) := \Phi^0(0,x)$$

their corresponding (compatible) initial values. Assume that H is a non trivial function of θ which means that

$$\partial_\theta H(x,\theta) \,=\, \partial_\theta r(x,\theta)\ {}^t\big(-1, f(\phi)\big) \not\equiv 0\,, \qquad r(x,\theta) := \mathbf{r}(0,x,\theta)\,.$$

Now, if $W^\varepsilon(x,\theta)$ is a profile which is periodic in θ and C^∞ with respect to $(\varepsilon,x,\theta)\in[0,1]\times\omega\times\mathbb{T}$, we know from the results of Section 2 that the perturbed initial data

$$\begin{aligned} \mathbf{u}^\varepsilon(0,x) = H^\varepsilon\big(x,\phi(x)/\varepsilon\big) &= H\big(x,\phi(x)/\varepsilon\big) + \varepsilon\, W^\varepsilon\big(x,\phi(x)/\varepsilon\big) \\ &= H\big(x,\phi(x)/\varepsilon\big) + \varepsilon\, W^0\big(x,\phi(x)/\varepsilon\big) + O(\varepsilon^2) \end{aligned} \tag{1.17}$$

is still compatible with the equation (1.2). This implies that, for all $\varepsilon\in\,]0,\varepsilon_0]$ with ε_0 small enough, the C^1 solution $\mathbf{u}^\varepsilon$ to the Cauchy problem (1.2)-(1.17) exists on the fixed domain Ω^τ.

The main question is:

What is on the domain Ω^τ the asymptotic behavior of the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$ when ε goes to zero ?

A classical approach would be to look for a solution $\mathbf{u}^\varepsilon$ of (1.2) of the form

$$\mathbf{u}^\varepsilon(t,x) \,=\, \mathbf{U}^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big) \tag{1.18}$$

where $\mathbf{U}^\varepsilon(t,x,\theta)$ is a smooth profile which is C^∞ with respect to the variables $(\varepsilon,t,x,\theta) \in [0,1] \times \Omega^\tau \times \mathbb{T}$. Replacing the expression (1.18) in the equation (1.2) leads to the following equation

$$\partial_t \mathbf{U}^\varepsilon \,+\, (\mathbf{U}^\varepsilon \cdot \nabla_x)\, \mathbf{U}^\varepsilon \,+\, \varepsilon^{-1}\, (\partial_t \Phi^0 + \mathbf{U}^\varepsilon \cdot \nabla_x \Phi^0)\, \partial_\theta \mathbf{U}^\varepsilon \,+\, \mathbf{f} \,=\, 0\,. \tag{1.19}$$

Of course $\mathbf{H}^0$ is a special solution of (1.19). Now, the linearized equation along $\mathbf{H}^0$ is

$$\partial_t \dot{\mathbf{U}}^\varepsilon + (\mathbf{H}^0 \cdot \nabla_x)\dot{\mathbf{U}}^\varepsilon \,+\, (\dot{\mathbf{U}}^\varepsilon \cdot \nabla_x)\mathbf{H}^0 \,+\, \varepsilon^{-1}\, (\dot{\mathbf{U}}^\varepsilon \cdot \nabla_x \Phi^0)\, \partial_\theta \mathbf{H}^0 \,=\, 0\,. \tag{1.20}$$

The factor ε^{-1} expresses the presence of a *singularity*. A usual way to derive energy estimates is to multiply (1.20) on the left by ${}^t\dot{\mathbf{U}}^\varepsilon$. But this method indicates that, at a fixed time $t > 0$, the solution $\dot{\mathbf{U}}^\varepsilon$ of (1.20) can be amplified by an exponential factor such as $e^{c\,t/\varepsilon}$ with $c > 0$. In this sense, the solution $\mathbf{H}^0$ of (1.19) is linearly unstable. The conclusion is that this first approach does not work.

Large amplitude oscillations on the velocity field, like $\mathbf{u}^\varepsilon$, are known to be strongly unstable. This observation has been noticed for over a century. It was already mentioned in works of Kelvin and Rayleigh. Since, it has motivated many studies till for instance the recent contributions of S. Friedlander, W. Strauss and M. Vishik (see [12] and the related references).

The amplification phenomena under question appear also in the field of nonlinear geometric optics [8]-[14]. There, they are often called up to explain why classical methods do not allow to construct non trivial perturbations $\mathbf{u}^\varepsilon$ of $\tilde{\mathbf{u}}^\varepsilon$, with $\mathbf{u}^\varepsilon$ defined on some open domain of determinacy $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ which does not shrink to the empty set as ε goes to zero.

In fact, the discussion related to the construction of a solution $\mathbf{u}^\varepsilon$ in the "proximity" of $\tilde{\mathbf{u}}^\varepsilon$ depends on the *size* of the perturbation and, technically, whether *variations* of the phase are allowed or not. In order to emphasize these different aspects, the arguments will be presented progressively, as indicated in the next picture.

	fixed phase Φ^0	variations of the phase
small amplitude perturbations of size O(ε)	instability	Section 4.1
large amplitude well polarized perturbations of size O(1)	Section 4.2	Section 4.3

Of course, Section 4.3 is concerned with the more general situation. It happens that it furnishes also the simpler and more direct approach. But we present it without appeal. First because Sections 4.1 and 4.2 have their own features and bring specific informations. Secondly because going through Sections 4.1 and 4.2 gives a good idea of the difficulties to overcome.

1.4.1 Perturbations of size $O(\varepsilon)$ with variations of the phase. In order to absorb the $O(\varepsilon)$ perturbation $\varepsilon\, W^\varepsilon$, a possible strategy is to consider the simple wave $(\mathbf{H}^0, \Phi^0)$ as a background state and to construct solutions which are *small* $O(\varepsilon)$ perturbations of $(\mathbf{H}^0, \Phi^0)$, that is

$$\begin{aligned} \mathbf{H}^\varepsilon(t,x,\tilde\theta) &= \mathbf{H}^0(t,x,\tilde\theta) + \varepsilon\, \mathbf{W}^\varepsilon(t,x,\tilde\theta)\,, \\ \Phi^\varepsilon(t,x,\theta) &= \Phi^0(t,x) + \varepsilon\, \mathbf{\Psi}^\varepsilon(t,x,\theta)\,, \end{aligned} \tag{1.21}$$

with the following rules of substitution

$$\mathbb{T} \ni \theta \mid \Phi^0(t,x)/\varepsilon\,, \qquad \mathbb{T} \ni \tilde\theta \mid \Phi^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big)/\varepsilon\,.$$

The introduction of the expression $\mathbf{\Psi}^\varepsilon$ is not surprising. Indeed, the oscillations $\{\mathbf{u}^\varepsilon\}_\varepsilon$ under study belong to a regime which is *supercritical* for (1.1) and also (at first sight) for (1.2) so that the classical separation between phase and amplitude is not sure to make sense. These two objects are linked together. This is obvious when looking at (1.12) or (1.16). By extension, when tackling the question of stability, it is natural to handle *simultaneously* perturbations on $\mathbf{H}$ and Φ. At the end of these manipulations, the exact solution $\mathbf{u}^\varepsilon$ of the original problem has the form

$$\begin{aligned} \mathbf{u}^\varepsilon(t,x) &= \mathbf{H}^\varepsilon\big(t,x,\Phi^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big)/\varepsilon\big) \\ &= \mathbf{H}^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon + \mathbf{\Psi}^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big)\big)\,. \end{aligned} \tag{1.22}$$

The status of the function $\mathbf{\Psi}^\varepsilon(t,x,\theta)$ is ambiguous. On the one hand, in (1.21), it is incorporated in the phase Φ^ε. On the other hand, at the level of (1.22), it plays the part of a phase shift so that it can also be considered as a contribution to the amplitude of the wave. Indeed, it suffices to introduce the profile amplitude

$$\mathbf{U}^\varepsilon(t,x,\theta) := \mathbf{H}^\varepsilon\big(t,x,\theta+\mathbf{\Psi}^\varepsilon(t,x,\theta)\big) \tag{1.23}$$

which leads to the expected representation (1.18). Observe that a $O(\varepsilon)$ perturbation of Φ^ε induces a $O(1)$ modification of $\mathbf{U}^\varepsilon$. Precisely, the implementation of $\mathbf{\Psi}^\varepsilon$ is the key to describe the propagation issued from the $O(\varepsilon)$ perturbation $\varepsilon\, W^\varepsilon$ by using *only* $O(\varepsilon)$ modifications of the profile $\mathbf{H}^\varepsilon$.

The transformation (1.21) corresponds to a *blow up* procedure since the state variable $\mathbf{u}\in\mathbb{R}^2$ is replaced by $(\mathbf{W},\mathbf{\Psi})\in\mathbb{R}^3$. It follows some overlap of unknowns in formula (1.23). For instance, at time $t=0$, it suffices to adjust the initial data

$$\tilde W^\varepsilon(x,\tilde\theta) \,:=\, \mathbf{W}^\varepsilon(0,x,\tilde\theta)\,, \qquad \Psi^\varepsilon(x,\theta)\,:=\,\mathbf{\Psi}^\varepsilon(0,x,\theta)$$

in such a way that

$$H\big(x,\theta+\Psi^\varepsilon(x,\theta)\big)+\varepsilon\,\tilde W^\varepsilon\big(x,\theta+\Psi^\varepsilon(x,\theta)\big) = H(x,\theta)+\varepsilon\,W^\varepsilon(x,\theta)\,. \tag{1.24}$$

There is some flexibility, a possible choice being

$$\Psi^\varepsilon(x,\theta)\,=\,0\,, \qquad \tilde W^\varepsilon(x,\tilde\theta)\,=\,W^\varepsilon(x,\tilde\theta)\,.$$

Note that a similar technique of blow up has already been employed by S. Alinhac (see [1] and the related references) in order to precise the life spans of solutions of two-dimensional quasilinear wave equations.

In our context, the fundamental point is that, to get a solution $\mathbf{U}^\varepsilon$ of (1.2) given by formula (1.22), it suffices to impose on the new unknown $(\mathbf{W}^\varepsilon,\mathbf{\Psi}^\varepsilon)\in\mathbb{R}^3$ a set of *two well posed quasilinear symmetric hyperbolic systems*, with coefficients smooth in $t,x,\theta,\tilde\theta$ and also $\varepsilon\in[0,1]$. The first system involves only $\mathbf{W}^\varepsilon(t,x,\tilde\theta)$ and writes

$$\begin{cases} \partial_t\mathbf{W}^\varepsilon+\big((\mathbf{H}^0+\varepsilon\,\mathbf{W}^\varepsilon)\cdot\nabla_x\big)\mathbf{W}^\varepsilon+(\mathbf{W}^\varepsilon\cdot\nabla_x)\mathbf{H}^0\,=\,0\,, \\ \mathbf{W}^\varepsilon(0,x,\tilde\theta)\,=\,\tilde W^\varepsilon(x,\tilde\theta)\,. \end{cases} \tag{1.25}$$

This is a Burgers type equation in (t,x) depending smoothly on the parameters $\varepsilon\in[0,1]$ and $\tilde\theta\in\mathbb{T}$ so that $\mathbf{W}^\varepsilon$ is a smooth profile uniquely determined as the local solution of this system (1.25).

The second equation concerns the unknown $\mathbf{\Psi}^\varepsilon(t,x,\theta)$. It is a quasi-linear equation in the variables (t,x,θ). More precisely, we find

$$
(1.26) \qquad \left\{ \begin{array}{l} \partial_t \mathbf{\Psi}^\varepsilon + \big((\mathbf{H}^\varepsilon(t,x,\theta+\mathbf{\Psi}^\varepsilon) \cdot \nabla_x \big) \mathbf{\Psi}^\varepsilon \\ \qquad + \big(\mathbf{W}^\varepsilon(t,x,\theta+\mathbf{\Psi}^\varepsilon) \cdot \nabla_x \big) \Phi^0 \; \partial_\theta \mathbf{\Psi}^\varepsilon \\ \qquad + \big(\mathbf{W}^\varepsilon(t,x,\theta+\mathbf{\Psi}^\varepsilon) \cdot \nabla_x \big) \Phi^0(t,x) \; = \; 0 \, , \\ \mathbf{\Psi}^\varepsilon(0,x,\theta) \; = \; \Psi^\varepsilon(x,\theta) \, . \end{array} \right.
$$

In (1.26) the function $\mathbf{H}^\varepsilon$ is $\mathbf{H}^0 + \varepsilon\, \mathbf{W}^\varepsilon$ where $\mathbf{W}^\varepsilon$ is the *fixed* function which has been previously determined by solving (1.25). Obviously, the Cauchy problem (1.26) is locally well posed uniformly in $\varepsilon \in [0,1]$. Hence the *nonlinear stability* becomes clear, as soon as it is understood in terms of the state variables $(\mathbf{W}^\varepsilon, \mathbf{\Psi}^\varepsilon)$.

Now, a simple Taylor expansion leads to the asymptotic expansion

$$
(1.27) \qquad \mathbf{u}^\varepsilon(t,x) = \mathbf{H}^0\big(t,x,\Phi^0(t,x)/\varepsilon + \mathbf{\Psi}^0\big(t,x,\Phi^0(t,x)/\varepsilon\big)\big) \; + \; O(\varepsilon)
$$

in $L^\infty(\Omega^\tau)$ as ε goes to 0 where $\mathbf{\Psi}^0$ is determined through the systems (1.25) and (1.26) with $\varepsilon = 0$, that is by first solving

$$
(1.28) \qquad \left\{ \begin{array}{l} \partial_t \mathbf{W}^0 + (\mathbf{H}^0 \cdot \nabla_x) \mathbf{W}^0 \; + \; (\mathbf{W}^0 \cdot \nabla_x) \mathbf{H}^0 \; = \; 0 \, , \\ \quad \mathbf{W}^0 \, (0,x,\tilde\theta) \; = \; \tilde W^0(x,\tilde\theta) \, , \end{array} \right.
$$

and then by looking at

$$
(1.29) \qquad \left\{ \begin{array}{l} \partial_t \mathbf{\Psi}^0 + \big(\mathbf{H}^0(t,x,\theta+\mathbf{\Psi}^0) \cdot \nabla_x \big) \mathbf{\Psi}^0 \\ \qquad + \big(\mathbf{W}^0(t,x,\theta+\mathbf{\Psi}^0) \cdot \nabla_x \big) \Phi^0 \; \partial_\theta \mathbf{\Psi}^0 \\ \qquad + \big(\mathbf{W}^0(t,x,\theta+\mathbf{\Psi}^0) \cdot \nabla_x \big) \Phi^0(t,x) \; = \; 0 \, , \\ \quad \mathbf{\Psi}^0 \, (0,x,\theta) \; = \; \Psi^0(x,\theta) \, . \end{array} \right.
$$

By the way, observe that the access to $\mathbf{\Psi}^0$ needs to identify $\mathbf{W}^0$. Thus, it depends on the initial data $\tilde W^0$ and Ψ^0, and therefore on W^0. Now, fix $t > 0$. The formula (1.23) shows explicitly that the *nonlinear* mapping

$$
\begin{array}{rcl} H^s_{loc}(\mathbb{R}^2 \times \mathbb{T}) & \longrightarrow & H^s_{loc}(\mathbb{R}^2 \times \mathbb{T}) \\ H^\varepsilon(x,\theta) & \longmapsto & \mathbf{U}^\varepsilon(t,x,\theta) \end{array} \quad , \qquad s \in \mathbb{R}
$$

from the profile Cauchy data (with respect to ϕ) to the profile solution $\mathbf{U}^\varepsilon(t,\cdot)$ $\big($with respect to $\Phi^0(t,\cdot)\big)$ is *not continuous*. Indeed, as ε goes to zero, the Cauchy data $H^\varepsilon(x,\theta)$ converges to $H(x,\theta)$. But the solution $\mathbf{U}^\varepsilon(t,x,\theta)$ converges to $\mathbf{U}^0(t,x,\theta) := \mathbf{H}^0\big(t,x,\theta + \mathbf{\Psi}^0(t,x,\theta)\big)$ where the function $\mathbf{\Psi}^0$ is in general non trivial and does not depend only on H (but also on W^0 !).

Again, this expresses instability features of (1.19). This instability can be removed once we allow variations of the phase. We will see in the next subsection 4.2 that it can also be gotten round by performing a change on the state variable $\mathbf{U}^\varepsilon$, a change which is singular in $\varepsilon \in]0,1]$.

In no way, this instability of the *linearized problem* can ruin the existence of the solution of the perturbed *nonlinear* Cauchy problem. Indeed, the solution exists, is smooth and is *uniformly bounded* as ε goes to 0 on a domain independent of $\varepsilon \in]0,1]$. This is the reason why we call it a *weak* instability. It is very interesting to observe that the nonlinear problem (1.1), when restricted to $\mathcal{V}_\iota^\tau(\omega)$, has a better behavior than the corresponding linearized problem which gives rise to exponentially growing up solutions as ε goes to 0.

1.4.2 Large amplitude well polarized modifications with fixed phase. The results of the previous section show that a perturbation of size $O(\varepsilon)$ at time $t = 0$ of the simple wave $\mathbf{H}^0(t,x,\Phi^0/\varepsilon)$ produces a solution of the form (1.18) where $\mathbf{U}^\varepsilon$ is given by the formula (1.23). Use a first order Taylor expansion in (1.23) to obtain

$$
\begin{aligned}
\mathbf{U}^\varepsilon(t,x,\theta) \,=\, & \mathbf{H}^0(t,x,\theta) \\
& + \boldsymbol{\Psi}^0(t,x,\theta) \, \int_0^1 \partial_\theta \mathbf{H}^0\big(t,x,\theta + s\,\boldsymbol{\Psi}^0(t,x,\theta)\big)\, ds + O(\varepsilon)\,.
\end{aligned}
\tag{1.30}
$$

Now, since

$$
\partial_\theta \mathbf{H}^0(t,x,\theta) \cdot \nabla \Phi^0(t,x) \,=\, 0\,, \qquad \forall\,(t,x,\theta) \in \Omega^\tau \times \mathbb{T}\,,
$$

the relation (1.30) can be interpreted as

$$
\mathbf{U}^\varepsilon(t,x,\theta) \,=\, \mathbf{H}^0(t,x,\theta) + \alpha(t,x,\theta)\; \nabla_x \Phi^0(t,x)^\perp + \varepsilon\, \mathbf{Z}^\varepsilon(t,x,\theta)\,,
\tag{1.31}
$$

where $\alpha(t,x,\theta)$ is a scalar smooth periodic profile and where $\mathbf{Z}^\varepsilon(t,x,\theta)$ is a smooth profile valued in $\mathbb{R}^2$. The formula (1.31) suggests to look for more general oscillating solutions (1.18) with a profile $\mathbf{U}^\varepsilon$ of the form

$$
\begin{aligned}
\mathbf{U}^\varepsilon(t,x,\theta) \,=\, & \mathbf{H}\big(t,x,\theta + \mathbf{V}^{\varepsilon 1}(t,x,\theta)\big) \\
& + \mathbf{V}^{\varepsilon 2}(t,x,\theta)\; \nabla_x \Phi^0(t,x)^\perp + \varepsilon\; \mathbf{V}^{\varepsilon 3}(t,x,\theta)\; \nabla_x \Phi^0(t,x)
\end{aligned}
\tag{1.32}
$$

including *large amplitude perturbations* in the direction of $\nabla_x \Phi^0(t,x)^\perp$. This corresponds again to a *blow up* of the state variables ($\mathbf{V}^{\varepsilon 1}$ plays the part of $\boldsymbol{\Psi}^\varepsilon$) which substitutes for $\mathbf{U}^\varepsilon$ the new unknown

$$
\mathbf{V}^\varepsilon(t,x,\theta) \,=\, {}^t(\mathbf{V}^{\varepsilon 1}, \mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3})(t,x,\theta) \,\in\, C^1(\Omega^\tau \times \mathbb{T}; \mathbb{R}^3)\,.
$$

The interest of this procedure becomes clear in Proposition 4.1. We find a well posed hyperbolic system for $\mathbf{V}^\varepsilon$.

More precisely, the singular equation (1.19) is exchanged with the Burger type equation (4.13) imposed on $\mathbf{V}^\varepsilon$. The advantage is that (4.13) involves coefficients which are smooth with respect to (t, x, θ) and also $\varepsilon \in [0, 1]$. Therefore, the construction of $\mathbf{V}^\varepsilon$ (and thereby $\mathbf{U}^\varepsilon$) on a domain $\Omega^\tau \times \mathbb{T}$ independent of ε can be achieved.

1.4.3 The two points of view conciliated. The subsection 4.2 gives a result (Proposition 4.1) which is more general than the one of subsection 4.1 (Theorem 4.1). Indeed, it allows to incorporate modifications of $\tilde{\mathbf{u}}^\varepsilon$ which are of size $O(1)$ in the direction $\nabla_x \Phi^0(t, x)^\perp$ instead of being of size $O(\varepsilon)$. Incidentally, this indicates that, eventually, passing through the construction of the simple wave $\tilde{\mathbf{u}}^\varepsilon$ is not necessary.

In fact, what is important is only the structures of $H_\perp(z)$ and $\Phi^0(t, x)$. Then, it suffices to conceive that the propagation of the oscillating part polarized according to $\nabla_x \Phi^0(t, x)^\perp$ (including $\mathbf{H}^*$ as an unknown!) is coupled with the $O(\varepsilon)$ terms. This observation is illustrated in subsection 4.3. There, the informations drawn from subsections 4.2 and 4.3 are compounded to propose a rapid and elegant version of our stability argument.

1.5 Applications.

When the oscillating initial data are well prepared for (1.1) which means that they satisfy

$$(1.33) \qquad \breve{\mathbf{u}}^\varepsilon(0, x) = \breve{h}^\varepsilon(x) = \breve{H}^\varepsilon\big(x, \phi(x)/\varepsilon\big) \in \mathcal{V}_0^\tau(\omega), \qquad \forall\, \varepsilon \in]0, 1]$$

the solution $\breve{\mathbf{u}}^\varepsilon(t, x)$ of (1.2)-(1.33) is also a solution of (1.1)-(1.33). By this way, the analysis of Section 4 brings informations related to the propagation of special singularities in Euler equations. This last Section lays stress on some consequences which can be thus extracted.

1.5.1 Various caustics phenomena for the Euler equations. The functions $\breve{\mathbf{u}}^\varepsilon(t, x)$ can also be interpreted as solutions of two dimensional compressible Euler equations and, in this interpretation, they are propagated along the derivation $\mathcal{L} := \partial_t + \mathbf{u} \cdot \nabla_x$. Now, in the usual classification, this vector field $\mathcal{L}$ is called linearly degenerate. However, the *first* phase Φ^0 or the *second* phase $\mathbf{\Psi}^0$ can develop shocks. The reason of this fact is that the nonlinearity of the transport equation enters into the eiconal equation through the coupling between the profiles and the phases. In the paragraph 5.1, we do not make a systematic study of such phenomena but, instead, we produce explicit examples.

1.5.2 Come back to Theorem 1.1. The objection to the concentration-cancellation property which is mentioned in Theorem 1.1 comes directly from the influence of $\mathbf{\Psi}^0$. The underlying mechanism is detailed in subsection 5.2. It exploits in a crucial manner the nonlinearity of ϕ (which is equivalent to the condition $f' \not\equiv 0$). Now, to incorporate nonlinear phases ϕ, we have to consider initial data $\{\breve{h}^\varepsilon\}_\varepsilon$ which do not give rise to simple waves (Lemma 3.3). Necessarily, such oscillations $\{\breve{h}^\varepsilon\}_\varepsilon$ contain non trivial terms of size $O(\varepsilon)$ or less. Under these conditions, the description of $\{\breve{\mathbf{u}}^\varepsilon\}_\varepsilon$ means to face the problem of stability in a regime which is *supercritical* for (1.1). This explains why the reply to the question raised in [2]-p. 479 is so delicate.

1.5.3 Interaction with small amplitude transversal waves. The blow up procedures of Section 4 have the effect to change the status of θ. Indeed, in the equation (1.19), the symbol θ stands for a *fast* variable whereas it becomes in (1.26) a *slow* variable. This observation gives the idea to apply at the level of (1.25) and (1.26) the standard results of weakly nonlinear geometric optics (see [13], [16] and [17]). The resulting analysis and the corresponding consequences are exposed in subsection 5.3.

In this introduction, we are satisfied with giving the idea of the underlying mechanisms. To this end, we fix a solution $\mathbf{W}^\varepsilon(t,x,\tilde{\theta})$ of (1.25) and we consider the solution $\mathbf{\Psi}^\varepsilon(t,x)$ of (1.26) apart. Of course, this manipulation does not really work since $\mathbf{W}^\varepsilon$ and $\mathbf{\Psi}^\varepsilon$ are linked together through the relation (1.24). We use this argument only to simplify below the presentation. The rigorous proof is given in chapter 5.3 (see Proposition 5.1).

Thus, we just argue here about the scalar conservation law (1.26). We consider the equation (1.26) in the neighbourhood of the basic solution $\mathbf{\Psi}^0(t,x,\theta)$ obtained by solving (1.29) with $\Psi^0 \equiv 0$. Nothing prevents $\mathbf{\Psi}^\varepsilon$ to still contain oscillations in x. In particular, we can seek solutions $\mathbf{\Psi}^\varepsilon$ of (1.26) which are small amplitude oscillations of the form

$$\mathbf{\Psi}^\varepsilon(t,x,\theta) \,=\, \mathbf{\Psi}^0(t,x,\theta) \,+\, \varepsilon\; \mathbf{\Psi}^{1,\varepsilon}\big(t,x,\theta,\zeta(t,x,\theta)/\varepsilon\big) \tag{1.34}$$

where the profile $\mathbf{\Psi}^{1,\varepsilon}(t,x,\theta,z) \in C^\infty(\Omega^\tau \times \mathbb{T} \times \mathbb{T};\mathbb{R})$ is smooth in $\varepsilon \in [0,1]$ and is assumed to verify

$$\partial_z \mathbf{\Psi}^{1,0}(0,x,\theta,z) \,\not\equiv\, 0\,. \tag{1.35}$$

The phase ζ must satisfy the eiconal equation

$$\begin{aligned}\partial_t \zeta + \big(\mathbf{H}^0(t,x,\theta+\mathbf{\Psi}^0)\cdot\nabla_x\big)\zeta \\ + \big(\mathbf{W}^0(t,x,\theta+\mathbf{\Psi}^0)\cdot\nabla_x\big)\Phi^0\ \partial_\theta\zeta \ = \ 0\,.\end{aligned}\tag{1.36}$$

We decide to complete (1.36) with some smooth initial data $\zeta_0 \in C_b^\infty(\omega;\mathbb{R})$ which does not depend on θ and which is transversal to ϕ. In other words

$$\zeta(0,x,\theta)=\zeta_0(x)\,,\qquad -\partial_1\zeta_0 + f(\phi)\ \partial_2\zeta_0 \not\equiv 0\,.\tag{1.37}$$

The equation (1.36) and the relation in (1.37) imply that

$$\partial_t(\partial_\theta\zeta)(0,x,\theta)\ =\ \partial_\theta r(x,\theta)\ \big(\partial_1\zeta_0 - f(\phi)\ \partial_2\zeta_0\big)\ \not\equiv\ 0$$

which means that ζ depends actually on θ when $t>0$ is small enough. Plug the expression $\mathbf{\Psi}^\varepsilon$ issued from (1.34) in (1.23) to find a solution $\mathbf{u}^\varepsilon$ of the equation (1.2) given by

$$\begin{aligned}\mathbf{u}^\varepsilon(t,x)\ =\ \mathbf{H}^\varepsilon\Big(\,t,x,\Phi^0(t,x)/\varepsilon + \mathbf{\Psi}^0\big(t,x,\Phi^0(t,x)/\varepsilon\big) \\ +\varepsilon\ \mathbf{\Psi}^{1,\varepsilon}\big(t,x,\Phi^0(t,x)/\varepsilon,\zeta\big(t,x,\Phi^0(t,x)/\varepsilon\big)/\varepsilon\Big)\,.\end{aligned}\tag{1.38}$$

Then, a first order Taylor expansion shows that

$$\begin{aligned}\mathbf{u}^\varepsilon(t,x)\ =\ \mathbf{U}^0\big(t,x,\Phi^0(t,x)/\varepsilon\big) \\ +\varepsilon\ \mathbf{G}^\varepsilon\Big(t,x,\Phi^0(t,x)/\varepsilon,\zeta\big(t,x,\Phi^0(t,x)/\varepsilon\big)/\varepsilon\Big)\end{aligned}\tag{1.39}$$

where $\mathbf{G}^\varepsilon(t,x,\theta,z)\in C^\infty(\Omega^\tau\times\mathbb{T}^2,\mathbb{R}^2)$ is smooth in $\varepsilon\in[0,1]$. Moreover, because of (1.35), it is subjected to $\partial_z G^0\not\equiv 0$ which means that, when $t>0$, the function $\mathbf{u}^\varepsilon(t,\cdot)$ does oscillate at the frequency ε^{-2}. On the other hand, at time $t=0$, the initial data $\mathbf{u}^\varepsilon(0,\cdot)$ involves only frequencies of size ε^{-1}. Indeed, in view of (1.37), the trace $\mathbf{u}^\varepsilon(0,\cdot)$ inherits the simpler form

$$\mathbf{u}^\varepsilon(0,x)\ =\ H\big(x,\phi(x)/\varepsilon\big)\ +\ \varepsilon\ \mathbf{G}^\varepsilon\big(0,x,\phi(x)/\varepsilon,\zeta_0(x)/\varepsilon\big)\,.\tag{1.40}$$

Thus, adding some transversal small amplitude oscillation in the Cauchy data, as in (1.40), produces a *composition* of the oscillations in the solution (1.39). In particular, the formula (1.39) shows that for all $t>0$ small enough and for some $\alpha\in\mathbb{N}^2\setminus\{(0,0)\}$, we have

$$\exists\, c\in\mathbb{R}_+^*\,;\qquad \sup_x\ |\partial_x^\alpha\mathbf{u}^\varepsilon(t,x)|\ \geq\ c\ \varepsilon^{1-2\,|\alpha|}\,.$$

On the contrary, at time $t=0$ and for the same $\alpha\in\mathbb{N}^2\setminus\{(0,0)\}$, we have

$$\exists\, C\in\mathbb{R}_+^*\,;\qquad \sup_x\ |\partial_x^\alpha\mathbf{u}^\varepsilon(t,x)|\ \leq\ C\ \varepsilon^{1-|\alpha|}\,.$$

The contrast between these two inequalities comes from the fact that the nonlinear interaction of the two waves of frequency $\sim (1/\varepsilon)$ produces waves with frequencies $\sim (1/\varepsilon^2)$.

This phenomenon presents an aspect which is completely different from the usual *resonance* of weakly non linear geometric optics. Indeed, in the case of resonances, the interaction of two (or more) waves having frequencies $\sim (1/\varepsilon)$ gives rise to the creation of new waves oscillating with the same size $O(1/\varepsilon)$ of frequency.

In fact, after a convenient change of the scaling, the above phenomenon could be regarded as resulting from a *superposition* (or composition) of weakly non-linear geometric optics (see the remark 5.3.4). The subsection 5.3 contains general examples of this sort. Also, it shows that such phenomena actually occur at the level of Euler equations. Thereby, it contains a *rigorous justification* that kinetic energy of solutions to (1.1) can be transferred from "low" wave numbers modes (namely of size ε^{-1}) to "high" wave numbers modes (namely of size ε^{-2}).

Such a transfer is apt to occur also at the level of the principal term of the oscillation (even if this subject will not be developed here). Moreover, it seems that the basic mechanism thus revealed could be repeated. It could be conceived as a starting point in order to study the propagation of more complicated oscillations, involving transfer of energy from large-scale motions to small-scale motions.

From a general point of view, all the subject of this paper falls under the scope of *supercritical* WKB analysis. This approach has been initiated by G. Lebeau [19]. Some small advancements have also been achieved in [5]-[6]-[7]. The articles [6]-[7] are specifically devoted to the propagation of oscillations which are solutions to (1.1). However, the situations studied in [6]-[7] differ from those under consideration here.

On the one hand, in [6]-[7], the data ϕ and H are selected arbitrarily leading generically to a *cascade of phases*. On the contrary, we adjust here the phase ϕ and the profile H so that such a cascade is avoided. This is a consequence of the relations (2.20) and (2.21) imposed on ϕ and H.

On the other hand, in [6]-[7], the WKB analysis is just formal or justified in the presence of a small anisotropic viscosity. On the contrary, our aim in the present paper is to face the problem of stability in the hyperbolic context. Concerning (1.1), this can be achieved here by restricting the choice of the initial data to the set $\mathcal{V}_0^{\tau}(\omega)$.

Contents

2 Well prepared initial data.

This Section is devoted to general considerations concerning the dynamics of a two dimensional vector field $\mathbf{u} = {}^t(\mathbf{u}^1, \mathbf{u}^2) \in \mathbb{R}^2$ described by the Burger equation (1.2). Let ω be a bounded open domain of $\mathbb{R}^2$ with closure $\bar{\omega}$. If necessary, make a translation in x to be sure that ${}^t(0,0) \in \omega$. Note C_b^k with $k \in \mathbb{N} \cup \{+\infty\}$ the space of functions with bounded continuous derivatives up to the order k. Select data h and $\mathbf{f}$ such that

$$\mathbf{u}(0,x) = h(x) \in C_b^1(\bar{\omega}), \qquad \mathbf{f}(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^2). \tag{2.1}$$

2.1 Preliminaries.

We start by looking at the life span of solutions to (1.2)-(2.1).

2.1.1 About the life span of solutions to Burger equations.

Look at the ordinary differential equation

$$\frac{d^2}{dt^2}\,\Xi(t,a,b)\;+\;\mathbf{f}\big(t,\Xi(t,a,b)\big)\;=\;0 \tag{2.2}$$

completed with

$$\Xi(0,a,b)\;=\;a\in\mathbb{R}^2\,,\qquad \tfrac{d}{dt}\,\Xi(0,a,b)\;=\;b\in\mathbb{R}^2\,.$$

Suppose that the source term $\mathbf{f}$ has at most a linear growth with respect to the space variable, uniformly in the time variable

$$\exists\, C\in\mathbb{R}_+\,;\qquad |\mathbf{f}(t,x)|\;\leq\;C\;(1+|x|)\,,\qquad \forall\,(t,x)\in\mathbb{R}\times\mathbb{R}^2\,. \tag{2.3}$$

Then, the application Ξ is globally defined and is smooth $\Xi\in C^1(\mathbb{R}\times\mathbb{R}^4)$. Introduce the graphs

$$G_h\;:=\;\big\{\,\big(x,h(x)\big)\,;\;x\in\bar\omega\,\big\}\;\subset\;\mathbb{R}^2\times\mathbb{R}^2\,,$$

$$DG_h\;:=\;\big\{\,\big(x,h(x),D_xh(x)\big)\,;\;x\in\bar\omega\,\big\}\;\subset\;\mathbb{R}^2\times\mathbb{R}^2\times\mathrm{M}^2(\mathbb{R})\,,$$

where $\mathrm{M}^d(\mathbb{R})$ is the space of $d\times d$ matrices with real coefficients.

Observe that G_h and DG_h are compact sets. The speed of propagation up to a time $T>0$ of a solution $\mathbf{u}$ to (1.2) coming from h and $\mathbf{f}$ is bounded by

$$c_{h,\mathbf{f}}(T)\;:=\|\;h\;\|_{L^\infty(\omega)}+T\;\sup\,\big\{\,|\mathbf{f}\big(t,\Xi(t,a,b)\big)|\,;\;(t,a,b)\in[0,T]\times G_h\,\big\}\,.$$

This allows to define the domain of determinacy

$$\Omega_{h,\mathbf{f}}(T)\;:=\;\big\{\,(t,x)\in[0,T[\times\mathbb{R}^2\,;\;B(x,t\,c_{h,\mathbf{f}}(T)[\,\subset\,\omega\,\big\}\;\not\equiv\;\emptyset$$

where $B(x,r[$ with $r>0$ is the open ball

$$B(x,r[:=\;\big\{\,\tilde x={}^t(\tilde x_1,\tilde x_2)\in\mathbb{R}^2\,;\;|\tilde x-x|^2=(\tilde x_1-x_1)^2+(\tilde x_2-x_2)^2<r^2\,\big\}\,.$$

Let $(t,a,b,M)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2\times\mathrm{M}^2(\mathbb{R})$. Introduce the 2×2 matrix

$$\Gamma(t,a,b,M)\;:=\;D_a\Xi(t,a,b)\;+\;D_b\Xi(t,a,b)\;M$$

and note

$$\mathbf{R}(a,b,M)\;:=\;\sup\,\big\{\,T\in[0,+\infty[\,;\;\det\Gamma(t,a,b,M)>0\,,\;\forall\,t\in[0,T[\,\big\}\,.$$

Lemma 2.1. *The Cauchy problem (1.2)-(2.1) has a C^1 solution* $\mathbf{u}$ *which is defined on the domain*

$$\Omega_{h,\mathbf{f}} \,:=\, \cup_{\{T;\, 0<T<T_{h,\mathbf{f}}\}}\, \Omega_{h,\mathbf{f}}(T)\,, \qquad T_{h,\mathbf{f}} \,:=\, \inf\, \big\{\, \mathbf{R}(z)\,;\; z \in DG_h \,\big\}\,.$$

Before giving the proof of this lemma, let us introduce a useful notation. Since Γ is a continuous function satisfying $\Gamma(0,\cdot) \equiv \mathrm{Id}$, we have $T_{h,\mathbf{f}} > 0$ so that $\Omega_{h,\mathbf{f}} \not\equiv \emptyset$. For any $\tau \in \,]0,1]$, define

$$\begin{aligned} \mathcal{B}^\tau(\omega) \,:=\, \big\{\, (h,\mathbf{f}) \in C^1_b(\omega) \times C^1(\mathbb{R}\times\mathbb{R}^2)\,;\; & |h(x)| \leq \tau^{-1}\,, \quad \forall\, x \in \bar\omega\,, \\ & \mathbf{R}\big(x, h(x), D_x h(x)\big) \in [\tau, +\infty]\,, \quad \forall\, x \in \bar\omega\,, \\ & |\mathbf{f}(t,x)| \leq \tau^{-1}\,\,(1+|x|)\,, \quad \forall\,(t,x) \in \mathbb{R}\times\bar\omega \,\big\}\,. \end{aligned}$$

It is clear that

$$\begin{aligned} & c^\tau \,:=\, \sup\, \big\{\, c_{h,\mathbf{f}}(T)\,;\; (h,\mathbf{f}) \in \mathcal{B}^\tau(\omega) \,\big\} \,<\, \infty\,, \\ & \tau \,\leq\, \inf\, \big\{\, T_{h,\mathbf{f}}\,;\; (h,\mathbf{f}) \in \mathcal{B}^\tau(\omega) \,\big\}\,. \end{aligned}$$

It follows that

$$\Omega^\tau \,:=\, \big\{\, (t,x) \in [0,\tau[\times\mathbb{R}^2\,;\; B(x, c^\tau\, t[\subset \omega \,\big\} \,\subset\, \Omega_{h,\mathbf{f}}\,.$$

By applying Lemma 2.1, we get:

Lemma 2.2. *The Cauchy problem (1.2)-(2.1) built with $(h,\mathbf{f}) \in \mathcal{B}^\tau(\omega)$ has a C^1 solution* $\mathbf{u}$ *which is defined on the domain Ω^τ.*

Proof of Lemma 2.1. Since unicity is clear in the context of C^1 functions, it remains to show that a solution $\mathbf{u}$ of (1.2)-(2.1) exists on $\Omega_{h,\mathbf{f}}(T)$ when $0 < T < T_{h,\mathbf{f}}$. Classical results yield a time $\tilde T > 0$ such that a solution $\mathbf{u}$ of (1.2)-(2.1) exists on $\Omega_{h,\mathbf{f}}(\tilde t)$ for all $\tilde t < \tilde T$. Take $\tilde T$ as large as possible.

If $\tilde T \geq T$, there is nothing to do. Thus, suppose that $\tilde T < T$. The continuation principle (exposed in Majda [20] paragraph 2.2) says that

$$\sup\;\; \big\{\, \|\, D_{\tilde x}\mathbf{u}(\tilde t, \tilde x)\,\|\,;\; (\tilde t,\tilde x) \in \Omega_{h,\mathbf{f}}(\tilde T) \,\big\} \,=\, +\infty\,. \tag{2.4}$$

To simplify the notations, define $\breve\Xi(t,x) := \Xi\big(t,x,h(x)\big)$ and

$$\breve\Gamma(t,x) \,:=\, \Gamma\big(t,x,h(x),D_x h(x)\big)\,, \qquad \breve R(x) \,:=\, \mathbf{R}\big(x,h(x),D_x h(x)\big)\,.$$

Since $\mathbf{u}$ can be integrated along characteristics, for each $(\tilde t,\tilde x) \in \Omega_{h,\mathbf{f}}(\tilde T)$, there is $x \in \omega$ such that

$$\mathbf{u}(\tilde t,\tilde x) \,=\, h(x) \,-\, \textstyle\int_0^{\tilde t}\, \mathbf{f}\big(t, \breve\Xi(t,x)\big)\,\, dt\,, \qquad \tilde x = \breve\Xi(\tilde t, x)\,. \tag{2.5}$$

It follows that

$$(2.6) \qquad D_{\tilde{x}}\mathbf{u}(\tilde{t},\tilde{x})\; \breve{\Gamma}(\tilde{t},x) \;=\; D_x h(x) \;-\; \textstyle\int_0^{\tilde{t}} D_x \mathbf{f}\big(t,\breve{\Xi}(t,x)\big)\; \breve{\Gamma}(t,x)\; dt\,.$$

On the one hand

$$\tilde{t} \,<\, \tilde{T} \,<\, T \,<\, T_{h,\mathbf{f}} \,\leq\, \breve{R}(x)\,, \qquad \forall\, x \in \omega\,.$$

On the other hand

$$\det \breve{\Gamma}(t,x) \,>\, 0\,, \qquad \forall\,(t,x) \in [0,\breve{R}(x)[\times\omega\,.$$

In particular

$$\det \breve{\Gamma}(t,x) \,>\, 0\,, \qquad \forall\,(t,x) \in [0,\tilde{T}[\times\omega\,.$$

Suppose that there is a sequence $\{(t_n,x_n)\}_n \in ([0,\tilde{T}[\times\omega)^{\mathbb{N}}$ which satisfies

$$\lim_{n\longrightarrow\infty}\; \breve{\Gamma}(t_n,x_n) \;=\; 0\,.$$

Extract a subsequence (given by $\ell : \mathbb{N} \longrightarrow \mathbb{N}$) such that

$$\lim_{n\longrightarrow\infty}\; \big(t_{\ell(n)}, x_{\ell(n)}, h(x_{\ell(n)}), D_x h(x_{\ell(n)})\big) \;=\; (\bar{t},\bar{z}) \in [0,\tilde{T}] \times DG_h\,.$$

The continuity of Γ with respect to the variables (t,a,b,M) guarantees that $\det \Gamma(\bar{t},\bar{z}) = 0$. This implies that

$$\mathbf{R}(\bar{z}) \,\leq\, \bar{t} \,\leq\, \tilde{T} \,<\, T_{h,\mathbf{f}} \,\leq\, \mathbf{R}(\bar{z})$$

which clearly is not possible. It means that

$$(2.7) \qquad \inf\;\; \big\{ \det \breve{\Gamma}(t,x)\,;\; (t,x) \in [0,\tilde{T}[\times\omega \,\big\} \;=\; c \,>\, 0\,.$$

Note $\mathrm{Co}\,(M)$ the co-matrix of M. Remember that

$$\breve{\Gamma}(\tilde{t},x)^{-1} \;=\; \big(\det \breve{\Gamma}(\tilde{t},x)\big)^{-1}\; \mathrm{Co}\;\big(\breve{\Gamma}(\tilde{t},x)\big)\,.$$

Use this and the relation (2.6) to get

$$\sup\;\; \big\{ \parallel D_{\tilde{x}}\mathbf{u}(\tilde{t},\tilde{x}) \parallel ;\; (\tilde{t},\tilde{x}) \in \Omega_{h,\mathbf{f}}(\tilde{T})\,\big\} \,<\, \infty$$

which is a contradiction with (2.4). Therefore $\tilde{T} \geq T$. □

Remark 2.1.2 - a more general situation. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^1 diffeomorphism. Look at the Burgers type equation

$$\partial_t \mathbf{u} \,+\, \big(g(\mathbf{u})\cdot\nabla_x\big)\mathbf{u} \;=\; 0\,.$$

In fact, this situation is equivalent to (1.2) with $\mathbf{f} \equiv 0$. Indeed, it suffices to take $g(\mathbf{u})$ as the new unknown. ◇

2.1.2 Three special cases.

This paragraph deals with

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \mathbf{f}_\iota = 0 , \qquad \mathbf{f}_\iota(t,x) := \iota\, x , \qquad \iota \in \{-1,0,1\} . \tag{2.8}$$

First, consider the case $\iota = 0$. When $\iota = 0$, simplifications occur in the preceding discussion. Introduce the domain of determinacy

$$\Omega_h := \big\{ (t,x) \in [0,+\infty[\times\mathbb{R}^2 ; \ B(x, t \parallel h \parallel_{L^\infty(\omega)} [\subset \omega \big\} .$$

Define the application

$$\begin{array}{rcll} I_h : \omega & \longrightarrow & \mathbb{R}^2 & \\ x & \longmapsto & I_h(x) = {}^t\big(I_h^1(x), I_h^2(x)\big) := {}^t\Big(\operatorname{div}_x h(x) \, , \, \det D_x h(x) \Big) . \end{array}$$

The image of ω by I_h is

$$I_h(\omega) := \big\{ I_h(x) ; \ x \in \omega \big\} .$$

Note $\bar I_h(\omega)$ the closure of $I_h(\omega)$. This is a compact set. Compute

$$R(a,b) := \sup \big\{ T \in \mathbb{R}_+ ; \ 1 + a\,t + b\,t^2 > 0 , \ \forall\, t \in [0,T[\big\} \in \mathbb{R}_+ \cup \{+\infty\} .$$

The result 2.3 below is true even if ω is not bounded.

Lemma 2.3. *Take $\iota = 0$. The Cauchy problem (2.8)-(2.1) has a C^1 solution $\mathbf{u}$ which is defined on the truncated cone*

$$\Omega_h(T_h) := \big\{ (t,x) \in \Omega_h ; \ t < T_h \big\} , \qquad T_h := \inf \big\{ R(y) ; \ y \in \bar I_h(\omega) \big\} .$$

Proof of Lemma 2.3. When ω is bounded, just apply the procedure of Lemma 2.1 to find $c_{h,\mathbf{f}_0}(T) = \parallel h \parallel_{L^\infty(\omega)}$ and

$$\Xi(t,a,b) = a + t\, b , \qquad \Gamma(t,a,b,M) = \mathrm{Id} + t\, M .$$

It follows that

$$\Omega_{h,\mathbf{f}_0}(T) \subset \Omega_{h,\mathbf{f}_0}(\tilde T) , \qquad \forall\, T \in]0, \tilde T[,$$

$$\det D_x \breve\Xi(t,x) = \det\big(\mathrm{Id} + t\, D_x h(x)\big) = 1 + I_h^1(x)\ t + I_h^2(x)\ t^2 .$$

Now, it is easy to see that $T_{h,\mathbf{f}_0} \equiv T_h$ and $\Omega_{h,\mathbf{f}_0} \equiv \Omega_h$.

When ω is not bounded, it suffices to remark that the time T_h^r associated with the restriction of h to the open domain $\omega \cap B(0,r[$ is such that

$$T_h \leq T_h^r , \qquad \forall\, r \in \mathbb{R}_+ , \qquad \lim_{r \longrightarrow +\infty} T_h^r = T_h .$$

Since the speed of propagation is bounded by $\parallel h \parallel_{L^\infty(\omega)}$, the expected result can be obtained by passing to the limit ($r \longrightarrow +\infty$). $\square$

Remark 2.1.3 - comparison with other results. Observe that

$$R^{-1}(+\infty) = \big\{(a,b) \in \mathbb{R}^2 \,;\, a \geq 0 \,,\, b \geq 0\big\} \cup \big\{(a,b) \in \mathbb{R}^2 \,;\, a < 0 \,,\, b \geq \tfrac{a^2}{4}\big\} \,.$$

Suppose that $\omega = \mathbb{R}^2$. Then, the solution of (2.8)-(2.1) with $\iota = 0$ is global in time if and only if

$$R\big(I_h(x)\big) \,=\, +\infty \,, \qquad \forall \, x \in \mathbb{R}^2 \,. \tag{2.9}$$

Simple computations indicate that this criterion is equivalent with the condition imposed in Theorem 2.2 of [21]. When (2.9) is violated, the preceding analysis of the local in time existence extends what is done in [21]. $\diamond$

Remark 2.1.4 - the other cases. When $\iota = -1$ or $\iota = 1$, the solutions of (2.2) are respectively

$$\Xi_{-1}(t,a,b) \,=\, a \,\mathrm{ch}\, t \,+\, b \,\mathrm{sh}\, t \,, \qquad \Xi_1(t,a,b) \,=\, a \,\cos t \,+\, b \,\sin t \,.$$

It follows that

$$\Gamma_{-1}(t,a,b,M) = \mathrm{ch}\, t \,(\mathrm{Id} + \mathrm{th}\, t \, M) \,, \quad \Gamma_1(t,a,b,M) = \cos t \,(\mathrm{Id} + \mathrm{tg}\, t \, M) \,.$$

Thus, the stopping times $T_{h,\mathbf{f}_\iota}$ which are associated with the data $(h, \mathbf{f}_\iota)$ are given by $T_{h,\mathbf{f}_\iota} = L_\iota(T_h)$ where $L_0(T_h) = T_h$ and

$$L_{-1}(T_h) \,=\, \begin{cases} \mathrm{argth}\, T_h & \text{if} \quad T_h < 1 \,, \\ +\infty & \text{if} \quad T_h \geq 1 \,, \end{cases} \qquad L_1(T_h) \,=\, \mathrm{arctg}\, T_h \,.$$

$\diamond$

Remark 2.1.5 - introduction of $\mathcal{B}^\tau_\iota(\omega)$. The condition (2.3) is obviously satisfied by the functions $\mathbf{f}_\iota$. Instead of $\mathcal{B}^\tau(\omega)$, c^τ and Ω^τ, we can consider

$$\begin{aligned} \mathcal{B}^\tau_\iota(\omega) \,&:=\, \big\{ \, h \in C^1_b(\omega; \mathbb{R}^2) \,;\; |h(x)| \leq \tau^{-1} \,, \quad \forall \, x \in \bar{\omega} \,, \\ &\qquad\qquad\qquad\qquad\qquad R\big(I_h(x)\big) \in [\tau, +\infty] \,, \quad \forall \, x \in \bar{\omega} \, \big\} \,, \\ c^\tau_\iota \,&:=\, \sup \, \big\{ \, c_{h,\mathbf{f}_\iota} \,;\; h \in \mathcal{B}^\tau_\iota(\omega) \, \big\} \,<\, \infty \,, \\ \Omega^\tau_\iota \,&:=\, \big\{ \, (t,x) \in [0, L_\iota(\tau)[\times \mathbb{R}^2 \,;\; B(x, c^\tau_\iota \, t[\subset \omega \, \big\} \,. \end{aligned}$$

If $h \in \mathcal{B}^\tau_\iota(\omega)$, a C^1 solution of (2.8)-(2.1) is defined on Ω^τ_ι. $\diamond$

2.1.3 Invariant sets given by Monge-Ampère equations.

The reason why the functions $\mathbf{f}_\iota$ have been distinguished is the following. When h is conveniently adjusted, the Cauchy problem (2.8)-(2.1) furnishes also a solution of (1.1)-(2.1). This fact is proved below. Observe that

$$\mathbf{f}_\iota(t,x) \,=\, \nabla_x \mathbf{p}_\iota(x) \,, \qquad \mathbf{p}_\iota(x) \,:=\, \iota \, \tfrac{1}{2} \, |x|^2 \,+\, C \,.$$

Therefore, the solutions of (2.8)-(2.1) which are incompressible

$$\mathrm{div}_x\, \mathbf{u}(t,x) \,=\, 0\,, \qquad \forall\,(t,x) \in \Omega_{h,\mathbf{f}_\iota} \tag{2.10}$$

can be interpreted as solutions of incompressible Euler equations. When $\iota = 0$, they satisfy (1.1) with $\mathbf{p}$ constant. Thus, they are also solutions of compressible Euler equations. In fact, they are subjected to the pressureless gas dynamics system.

Suppose to simplify the discussion that ω is simply connected in $\mathbb{R}^2$. Then, look at the nonlinear functional set

$$\mathcal{V}_\iota^\tau(\omega) \,:=\, \big\{\, h = {}^t(-\partial_2\varrho, \partial_1\varrho) \in C^1_b(\omega;\mathbb{R}^2)\,;\; |h(x)| \le \tau^{-1}\,, \\ \varrho \in C^2(\omega;\mathbb{R})\,,\; \det\, D^2_{xx}\,\varrho \equiv \iota \,\big\}\,.$$

According to the usual terminology, this definition involves a *Monge-Ampère* equation (on ϱ) which is called hyperbolic, degenerate or elliptic when respectively $\iota = -1$, $\iota = 0$ or $\iota = 1$. Obviously, the subset $\mathcal{V}_\iota^\tau(\omega)$ is not empty.

Lemma 2.4. *The C^1 solution $\mathbf{u}$ of the Cauchy problem (2.8)-(2.1) is subjected to (2.10) if and only if $h \in \mathcal{V}_\iota^\tau(\omega)$ for some $\tau \in]0,1]$.*

Remark 2.1.6 - another interpretation of $\mathcal{V}_\iota^\tau(\omega)$. Introduce

$$\tilde{\mathcal{V}}_\iota^\tau(\omega) \,:=\, \big\{\, h \in C^1_b(\omega;\mathbb{R}^2)\,;\; |h(x)| \le \tau^{-1}\,,\; I_h(x) = {}^t(0,\iota)\,,\; \forall\, x \in \omega \,\big\}\,.$$

Let $h = {}^t(-\partial_2\varrho, \partial_1\varrho) \in \mathcal{V}_\iota^\tau(\omega)$. Obviously

$$I_h^1(x) \,=\, 0\,, \qquad I_h^2(x) \,=\, \det D^2_{xx}\varrho(x) \,=\, \iota\,, \qquad R\big(I_h(x)\big) \,\ge\, 1\,, \qquad \forall\, x \in \omega\,.$$

It follows that

$$\mathcal{V}_\iota^\tau(\omega) \,\subset\, \tilde{\mathcal{V}}_\iota^\tau(\omega) \,\subset\, \mathcal{B}_\iota^1(\omega)\,, \qquad \forall\, \tau \in]0,1]\,.$$

Now, if $h \in \tilde{\mathcal{V}}_\iota^\tau(\omega)$, the restriction $I_h^1 \equiv 0$ means that h coincide with ${}^t(-\partial_2\varrho, \partial_1\varrho)$ for some scalar function $\varrho \in C^2(\omega;\mathbb{R})$. Then, the condition $I_h^2 \equiv \iota$ is equivalent to $\det D^2_{xx}\,\varrho \equiv \iota$. Therefore $\tilde{\mathcal{V}}_\iota^\tau(\omega) \equiv \mathcal{V}_\iota^\tau(\omega)$.

If $h \in \mathcal{V}_0^\tau(\mathbb{R}^2)$, the solution $\mathbf{u}$ of (1.2)-(2.1) or (1.1)-(2.1) is global in time. Moreover, the set $\mathcal{V}_0^\tau(\mathbb{R}^2)$ is invariant under the flow

$$h \in \mathcal{V}_0^\tau(\mathbb{R}^2) \quad \Longrightarrow \quad \mathbf{u}(t,\cdot) \in \mathcal{V}_0^\tau(\mathbb{R}^2)\,, \qquad \forall\, t \in \mathbb{R}_+\,. \qquad \diamond$$

Proof of Lemma 2.4. When $\iota = 0$, the result 2.4 can be deduced from Theorem 2.6 in [9]. The proof given below is different and more direct. It is also more general since it allows to incorporate the cases $\iota = \pm 1$.

$\circ$ Suppose that $\mathbf{u}$ is a solution of (2.8)-(2.1) which is subjected to (2.10). On the one hand, the relation (2.10) means that the two (complex) eigenvalues λ_1 and λ_2 of the matrix $D_x\mathbf{u}(t,x)$ are opposite, say $\lambda_1 = -\lambda_2 = \lambda$. On the other hand, the equation (2.8) implies that

$$(\partial_t + \mathbf{u}\cdot\nabla_x)\, D_x\mathbf{u} \,+\, (D_x\mathbf{u})^2 \,+\, \iota\ \mathrm{Id} \,=\, 0\,, \qquad \forall\,(t,x)\in\Omega_{h,\mathbf{f}_\iota}\,.$$

Take the trace to get

$$\mathrm{Tr}\,(D_x\mathbf{u})^2 \,+\, 2\ \iota \,=\, 2\ (\lambda^2+\iota) \,=\, 0\,, \qquad \forall\,(t,x)\in\Omega_{h,\mathbf{f}_\iota}\,.$$

Look at what happens when $t = 0$. Since ω is supposed to be simply connected in $\mathbb{R}^2$, there is $\varrho\in C^2(\omega;\mathbb{R})$ such that

$$\mathbf{u}(0,x) \,=\, h(x) \,=\, {}^t\nabla_x\varrho(x)^\perp \,:=\, {}^t\big(-\partial_2\varrho(x),\partial_1\varrho(x)\big)\,, \qquad \forall\,x\in\omega$$

and the preceding condition reduces to

$$-\,\lambda(0,x)^2 \,=\, \det D_x\mathbf{u}(0,x) \,=\, \det D_x h(x) \,=\, \det D^2_{xx}\varrho(x) \,=\, \iota\,.$$

Take τ small enough to be sure that $h\in\mathcal{V}^\tau_\iota(\omega)$.

$\circ$ Conversely, suppose that $h\in\mathcal{V}^\tau_\iota(\omega)$. Apply Lemma 2.1 to find a solution $\mathbf{u}$ of (2.8)-(2.1) on $\Omega_{h,\mathbf{f}_\iota}$. Note λ_1 and λ_2 the two eigenvalues of $D_x\mathbf{u}$. Deduce from (2.8) that

$$(\partial_t + \mathbf{u}\cdot\nabla_x)\,(D_x\mathbf{u})^2 \,+\, 2\ (D_x\mathbf{u})^3 \,+\, 2\ \iota\ D_x\mathbf{u} \,=\, 0\,, \qquad \forall\,(t,x)\in\Omega_{h,\mathbf{f}_\iota}\,.$$

Observe that

$$\mathrm{Tr}\,(D_x\mathbf{u})^3 \,=\, \lambda_1^3 \,+\, \lambda_2^3 \,=\, (\lambda_1+\lambda_2)\ (\lambda_1^2-\lambda_1\ \lambda_2+\lambda_2^2)\,.$$

It follows that

$$\left\{\begin{array}{l} (\partial_t + \mathbf{u}\cdot\nabla_x)\,\mathrm{div}_x\,\mathbf{u} \,+\, \big[\mathrm{Tr}\,(D_x\mathbf{u})^2+2\,\iota\big] \,=\, 0\,, \\ (\partial_t + \mathbf{u}\cdot\nabla_x)\,\big[\mathrm{Tr}\,(D_x\mathbf{u})^2+2\,\iota\big] \,+\, 2\ (\lambda_1^2-\lambda_1\ \lambda_2+\lambda_2^2+\iota)\ \mathrm{div}_x\,\mathbf{u} \,=\, 0\,. \end{array}\right.$$

By hypothesis, at time $t=0$, we start with

$$\mathrm{div}_x\, h(x) \,=\, 0\,, \qquad \mathrm{Tr}\, D_x h(x)^2 \,+\, 2\,\iota \,=\, 2\ (-\det D^2_{xx}\varrho(x) \,+\, \iota) \,=\, 0\,.$$

Observe that

$$\big\{\,(\partial_t + \mathbf{u}\cdot\nabla_x)\,\mathbf{Z} = B\,\mathbf{Z}\,, \quad \mathbf{Z}(0,\cdot)\equiv 0\,, \quad \mathbf{Z} := {}^t\big(\mathrm{div}_x\,\mathbf{u}, \mathrm{Tr}\,(D_x\mathbf{u})^2+2\,\iota\big)$$

where $B(t,x)$ is a continuous and bounded function. Applying Gronwall's Lemma gives rise to $\mathbf{Z}\equiv 0$, that is

$$\mathrm{div}_x\,\mathbf{u}(t,x) \,=\, 0\,, \qquad \mathrm{Tr}\, D_x\mathbf{u}(t,x)^2 \,+\, 2\,\iota \,=\, 0\,, \qquad \forall\,(t,x)\in\Omega_{h,\mathbf{f}_\iota}\,.$$

In particular, the divergence free condition (2.10) is verified. $\square$

Remark 2.1.7 - about the choice of ι. In the preceding proof, the value of ι can be fixed arbitrarily in $\mathbb{R}$. We take $\iota \in \{-1, 0, 1\}$ because, when $\iota \in \mathbb{R}^*$, the change of time-space variables $(t, x) \, / \, |\iota|^{-1/2} \, (t, x)$ reduces to the situation $\iota \in \{-1, 1\}$. In fact, what is important in the demonstration is that ι does not depend on (t, x). $\diamond$

2.1.4 Large amplitude monophase oscillations.

According to the preceding construction, solutions $\mathbf{u}$ of (2.8)-(2.1) issued from $h \in \mathcal{B}_\iota^\tau(\omega)$ are defined on Ω^τ. Now, the constraint $h \in \mathcal{B}_\iota^\tau(\omega)$ does not imply a bound on all derivatives contained in $D_x h$. In particular, with τ fixed, it is possible to find families $\{h^\varepsilon\}_{\varepsilon \in \,]0,1]}$ which satisfy

$$(2.11) \qquad h^\varepsilon \in \mathcal{B}_\iota^\tau(\omega) \, , \qquad \forall \, \varepsilon \in]0, 1] \, ,$$

or

$$(2.12) \qquad h^\varepsilon \in \mathcal{V}_\iota^\tau(\omega) \, , \qquad \forall \, \varepsilon \in]0, 1] \, .$$

but whose derivatives $\{D_x h^\varepsilon\}_\varepsilon$ are not uniformly bounded. For instance, we can seek

$$(2.13) \qquad \lim_{\varepsilon \longrightarrow 0} \quad \| \, D_x h^\varepsilon \, \|_{L^2(\omega)} = \, +\infty \, .$$

Passing to the weak L^2-limit in (2.11) or (2.12) allows to capture the weak L^2-closures $\bar{\mathcal{B}}_\iota^\tau(\omega)$ and $\bar{\mathcal{V}}_\iota^\tau(\omega)$ of respectively $\mathcal{B}_\iota^\tau(\omega)$ and $\mathcal{V}_\iota^\tau(\omega)$. Since the functional sets $\mathcal{B}_\iota^\tau(\omega)$ and $\mathcal{V}_\iota^\tau(\omega)$ are nonlinear (their definitions involve the computation of $\det D_x h$), the sets $\bar{\mathcal{B}}_\iota^\tau(\omega)$ and $\bar{\mathcal{V}}_\iota^\tau(\omega)$ can be much bigger than $\mathcal{B}_\iota^\tau(\omega)$ and $\mathcal{V}_\iota^\tau(\omega)$.

This is precisely through the flexibility of the selected weak L^2-topology that complicated phenomena occurring at the level of Euler equations can be investigated even if the initial data h^ε are very constrained.

Now, the idea is to perform a nonlinear geometric optics under constraint, the constraint being given by $\mathcal{B}_\iota^\tau(\omega)$ or $\mathcal{V}_\iota^\tau(\omega)$. To put this in concrete form, we need to consider families $\{h^\varepsilon\}_\varepsilon$ having a specific behaviour as ε goes to zero. Some material is needed to describe the preliminary oscillating structure of h^ε. Fix $J \in \mathbb{N}$. Introduce :

2.a) a phase $\phi \in C^\infty(\omega; \mathbb{R})$ with $\nabla_x \phi \in C_b^\infty(\omega; \mathbb{R})$. Assume that ϕ is not stationary. More precisely, impose

$$(2.14) \qquad \exists \, c > 0 \, , \qquad \partial_2 \phi(x) \, \geq \, c \, , \qquad \forall \, x \in \omega \, ,$$

2.b) a profile $H \equiv H_{-1} = {}^t(H^1, H^2) \in C_b^\infty(\omega \times \mathbb{T}; \mathbb{R}^2)$ which is non trivial

$$(2.15) \qquad \exists\,(x,\theta) \in \omega \times \mathbb{T}\,; \qquad \partial_\theta H(x,\theta) \neq 0\,,$$

2.c) other profiles

$$H_j = {}^t(H_j^1, H_j^2) \in C_b^\infty(\omega \times \mathbb{T}; \mathbb{R}^2)\,, \qquad j \in \{0, \cdots, J-1\}\,,$$

2.d) a function $rh^\varepsilon \in C_b^\infty(\omega; \mathbb{R}^2)$ which is controlled by

$$(2.16) \qquad \sup\,\big\{\, \| \, rh^\varepsilon \, \|_{L^\infty(\omega)} \; + \varepsilon^{J-1} \;\; \| \, D_x \, rh^\varepsilon \, \|_{L^\infty(\omega)}\,;\; \varepsilon \in]0,1] \,\big\} \, < \, \infty\,.$$

With all these ingredients, build the asymptotic expansion

$$(2.17) \qquad h^\varepsilon(x) \, = \, \textstyle\sum_{j=-1}^{J-1} \, \varepsilon^{j+1} \; H_j\big(x, \phi(x)/\varepsilon\big) \, + \, \varepsilon^{J+1} \; rh^\varepsilon(x)\,.$$

Observe that h^ε is a large amplitude oscillating wave

$$h^\varepsilon(x) \, = \, H\big(x, \phi(x)/\varepsilon\big) \, + \, O(\varepsilon)\,, \qquad \partial_\theta H \, \not\equiv \, 0\,.$$

In the next subsection, we identify necessary and sufficient constraints to impose on ϕ and the H_j in order to have (2.11) for some fixed $\tau \in]0,1]$.

2.2 Well prepared families for the Burger equation.

The hypothesis 2.a), $\cdots$, 2.d) guarantee (2.13) and the fact that

$$\exists\, \tau \in \mathbb{R}_+\,; \qquad \sup\,\big\{\, |h^\varepsilon(x)|\,;\; (\varepsilon, x) \in]0,1] \times \omega \,\big\} \, \leq \, \tau^{-1} \, < \, \infty\,.$$

Now, to obtain a family $\{h^\varepsilon\}_\varepsilon \in \mathcal{B}_\iota^\tau(\omega)^{]0,1]}$ with corresponding solutions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ defined on the domain Ω_ι^τ (which does not shrink to the empty set as ε goes to 0), it suffices to check the condition on I_{h^ε} which is stated below.

Definition 2.1. *We say that the family $\{h^\varepsilon\}_\varepsilon$, where h^ε is defined as in (2.17) and is made of ingredients satisfying 2.a), $\cdots$, 2.d), is well prepared on ω for the Burger equation (1.2) if*

$$(2.18) \qquad \exists\, \tau \in \mathbb{R}_+\,; \qquad R\big(I_{h^\varepsilon}(x)\big) \in [\tau, +\infty]\,, \qquad \forall\,(x,\varepsilon) \in \omega \times]0,1]\,.$$

Any function $u \in L^1(\omega \times \mathbb{T})$ can be decomposed according to

$$u(x,\theta) \, = \, \langle u \rangle(x) + u^*(x,\theta) \, = \, \bar{u}(x) + u^*(x,\theta)$$

where $\langle u \rangle \equiv \bar{u}$ is the mean value

$$\langle u \rangle(x) \, = \, \bar{u}(x) \, := \, \textstyle\int_{\mathbb{T}} \, u(x,\theta) \; d\theta\,.$$

Introduce the oscillating support of u which is

$$\text{osup } u \,:=\, \big\{\, x\in\omega\,;\ u^*(x,\cdot)\not\equiv 0 \,\big\}\,.$$

Define also

$$(\partial_\theta^{-1} u^*)(x,\theta) \,:=\, \int_0^\theta u^*(x,\tilde\theta)\ d\tilde\theta \,-\, \int_0^1 \Big(\int_0^\theta u^*(x,\tilde\theta)\ d\tilde\theta \Big)\ d\theta\,.$$

In view of (2.14), for each $x_1\in\mathbb{R}$, the application

$$\begin{array}{rcl} \omega(x_1) \,:=\, \big\{\, x_2\,;\ (x_1,x_2)\in\omega \,\big\} & \longrightarrow & \mathbb{R} \\ x_2 & \longmapsto & \phi(x_1,x_2) \end{array}$$

is strictly increasing. The image set

$$\mathcal{I}(x_1) \,:=\, \big\{\, \phi(x_1,x_2)\,;\ x_2\in\omega(x_1) \,\big\}$$

is open. Introduce the other open set

$$\mathcal{I} \,:=\, \cup_{x_1\in\mathbb{R}}\, \mathcal{I}(x_1) \,\subset\, \mathbb{R}\,.$$

Seek a family $\{h^\varepsilon\}_\varepsilon$ which is well prepared on ω for (1.2). Easy computations indicate that the analysis can be reduced to the case $\omega = \text{osup}\, H$. Thus, from now on, we can suppose that $\omega=\text{osup}\, H^*$. We suppose moreover that the curve $\{x\in\omega\,;\,\phi(x)=z\}$ is connected for all $z\in\mathcal{I}$. These are technical assumptions which simplify the following statements.

Lemma 2.5. *Under the assumptions mentioned above, the family* $\{h^\varepsilon\}_\varepsilon$ *is subjected to the condition (2.18) if and only if* (H,ϕ) *satisfy the three following conditions:*

i) the profile H^* *is polarized according to*

$$\nabla_x\phi(x)\cdot H^*(x,\theta) \,=\, 0\,, \qquad \forall\,(x,\theta)\in\omega\times\mathbb{T}\,. \tag{2.19}$$

ii) there exists f *in* $C_b^\infty(\mathcal{I};\mathbb{R})$ *such that*

$$\partial_1\phi(x) \,=\, f\big(\phi(x)\big)\ \partial_2\phi(x)\,, \qquad \forall\, x\in\omega\,. \tag{2.20}$$

iii) there exists g *in* $C_b^\infty(\mathcal{I};\mathbb{R})$ *such that*

$$f\big(\phi(x)\big)\ \bar H^1(x) \,+\, \bar H^2(x) \,=\, g\big(\phi(x)\big)\,, \qquad \forall\, x\in\omega\,. \tag{2.21}$$

Couples (H,ϕ) which satisfy (2.19), (2.20) and (2.21) for some f and g are called *compatible* with (1.2).

Proof of Lemma 2.5. Note that

$$R(a,b) \,:=\, \begin{cases} +\infty & \text{if} \quad a<0 \quad \text{and} \quad b>\frac14\, a^2\,, \\ +\infty & \text{if} \quad a\geq 0 \quad \text{and} \quad b\geq 0\,, \\ -\frac1a & \text{if} \quad a<0 \quad \text{and} \quad b=0\,. \end{cases}$$

For all other values of (a,b), one has

$$R(a,b) \,=\, -\,\frac{a}{2\,b} \,-\, \frac{(a^2-4\,b)^{\frac{1}{2}}}{2\,b}\,.$$

It follows that

$$\begin{aligned} R^{-1}([\tau,+\infty]) \,:=\, & \big\{\,(a,b)\in\mathbb{R}^2\,;\ a\,\leq\,-\,2\,\tau^{-1}\,,\ b\,>\,\tfrac{1}{4}\,a^2\,\big\} \\ & \cup\big\{\,(a,b)\in\mathbb{R}^2\,;\ a\,\geq\,-\,2\,\tau^{-1}\,,\ b\,\geq\,-\,\tau^{-2}\,(\tau\,a+1)\,\big\}\,. \end{aligned}$$

∘ Suppose that $\{h^\varepsilon\}_\varepsilon$ is subjected to (2.18). Observe that

$$I_{h^\varepsilon}(x) = \varepsilon^{-1}\ I_{-1}\big(x,\varepsilon^{-1}\,\phi(x)\big)\,+\,O(1)\,,\qquad I_{-1}(x,\theta) = {}^t(I^1_{-1},I^2_{-1})(x,\theta)$$

with $I^1_{-1} = \nabla_x\phi\cdot\partial_\theta H^*$ and

$$I^2_{-1} \,=\, \partial_1\phi\,(\partial_\theta H^1\,\partial_2 H^2 - \partial_\theta H^2\,\partial_2 H^1) + \partial_2\phi\,(\partial_\theta H^2\,\partial_1 H^1 - \partial_\theta H^1\,\partial_1 H^2)\,.$$

Select any $(x,\theta)\in\omega\times\mathbb{T}$ such that $\phi(x)\neq 0$. Introduce

$$\varepsilon_k \,:=\, \phi(x)\ (\theta+\iota\,k)^{-1}\,,\qquad \iota\,:=\,\mathrm{sgn}\ \phi(x)\,,\qquad k\in\mathbb{N}\setminus\{0,1\}\,.$$

By construction

$$\text{(2.22)}\qquad I_{h^{\varepsilon_k}}(x) \,=\, \varepsilon_k^{-1}\ I_{-1}(x,\theta)\,+\,O(1)\,.$$

This indicates that the sequence $\big\{I_{h^{\varepsilon_k}}(x)\big\}_k$ is asymptotic when k goes to ∞ with the half line

$$\mathcal{D} \,:=\, \big\{\,\lambda\ I_{-1}(x,\theta)\,;\ \lambda\in\mathbb{R}_+\,\big\}\,.$$

In view of the geometry of the set $R^{-1}([\tau,+\infty])$, this is compatible with (2.18) only if $I^1_{-1}(x,\theta)\,\geq\,0$. Because of (2.14) and the continuity of the function I^1_{-1}, it leads to

$$I^1_{-1}(x,\theta)\,\geq\,0\,,\qquad \forall\,(x,\theta)\in\omega\times\mathbb{T}\,.$$

The foregoing shows that I^1_{-1} is a positive function. On the other hand, it is obviously a periodic function with mean zero. Therefore, it must be zero which is exactly the polarization condition (2.19). There is some scalar function $s\equiv s^*\in C^\infty(\omega\times\mathbb{T})$ which is such that

$$\text{(2.23)}\qquad H(x,\theta) \,=\, \bar{H}(x)\,+\,s^*(x,\theta)\ \nabla_x\phi(x)^\perp\,,\qquad \nabla_x\phi^\perp \,:=\, \begin{pmatrix} -\partial_2\phi \\ \partial_1\phi \end{pmatrix}\,.$$

The information (2.23) allows to simplify the expression of

$$\begin{aligned} I^2_{-1} \,=\, & \big[\,\partial_1\phi\ \partial_2\phi\ (\partial_1\bar{H}^1 - \partial_2\bar{H}^2)\,-\,(\partial_1\phi)^2\ \partial_2\bar{H}^1\,+\,(\partial_2\phi)^2\ \partial_1\bar{H}^2\,\big]\ \partial_\theta s^* \\ & +\big[\,-\,2\ \partial_1\phi\ \partial_2\phi\ \partial^2_{12}\phi\,+\,(\partial_1\phi)^2\ \partial^2_{22}\phi\,+\,(\partial_2\phi)^2\ \partial^2_{11}\phi\,\big]\ s^*\ \partial_\theta s^*\,. \end{aligned}$$

Since now $I^1_{-1}(x,\theta)=0$, the sequence $\{I_{h^{\varepsilon_k}}(x)\}_k$ is asymptotic when k goes to ∞ with

$$\mathcal{D}_\pm := \big\{ (0,b)\,;\ \pm b>0 \big\} \quad \text{if} \quad \pm\, I^2_{-1}(x,\theta) \,>\, 0\,.$$

In view of the geometry of the set $R^{-1}([\tau,+\infty])$, this is compatible with (2.18) on condition that $I^2_{-1}(x,\theta)\geq 0$. Because of (2.14) and the continuity of the function I^2_{-1}, it leads to

$$I^2_{-1}(x,\theta) \,\geq\, 0\,, \qquad \forall\,(x,\theta)\in\omega\times\mathbb{T}\,.$$

Again I^2_{-1} is a positive function. It is periodic and the formula given above indicates that it is with mean zero. Therefore, it must be zero. Since by hypothesis $\omega=\operatorname{osup} s^*$, the condition $I^2_{-1}\equiv 0$ amounts to the same thing as imposing for all $x\in\omega$ the two relations

$$\partial_1\phi\ \partial_2\phi\ (\partial_1\bar{H}^1-\partial_2\bar{H}^2)\,-\,(\partial_1\phi)^2\ \partial_2\bar{H}^1\,+\,(\partial_2\phi)^2\ \partial_1\bar{H}^2 \,=\, 0\,, \tag{2.24}$$

$$2\ \partial_1\phi\ \partial_2\phi\ \partial^2_{12}\phi\,-\,(\partial_1\phi)^2\ \partial^2_{22}\phi\,-\,(\partial_2\phi)^2\ \partial^2_{11}\phi \,=\, 0\,. \tag{2.25}$$

In view of (2.14), the identity (2.25) is equivalent to

$$(-\partial_2\phi\ \partial_1\,+\,\partial_1\phi\ \partial_2)\,(\partial_1\phi/\partial_2\phi)\,=\,0\,, \qquad \forall\,x\in\omega\,. \tag{2.26}$$

This differential equation implies that the quotient $\partial_1\phi/\partial_2\phi$ is locally constant on each level curve of ϕ. In fact, it is constant on the whole curve because by hypothesis the curve is connected. Since moreover the application $x_2\longmapsto\phi(x_1,x_2)$ is a local C^∞ diffeomorphism, the property (2.20) must be achieved for some $f\in C^\infty_b(\mathcal{I};\mathbb{R})$.

Since ϕ is now subjected to the restriction (2.20), the relation (2.24) can be simplified according to

$$(-\partial_2\phi\ \partial_1\,+\,\partial_1\phi\ \partial_2)\,\big(\bar{H}^2+f(\phi)\,\bar{H}^1\big)\,=\,0\,. \tag{2.27}$$

The same argument as above gives (2.21).

$\circ$ Conversely, choose any couple $(\bar{H},\phi)$ satisfying (2.20) and (2.21). Select any profile s^*. Define H as in (2.23). It implies that $I_{-1}\equiv 0$ so that

$$\exists\,C>0\,; \qquad |I^1_{h^\varepsilon}(x)|\,\leq\,C\,, \quad |I^2_{h^\varepsilon}(x)|\,\leq\,C\,, \qquad \forall\,(\varepsilon,x)\in\,]0,1]\times\omega\,.$$

In particular, we find (2.18) for some $\tau\in\,]0,1]$. $\square$

Remark 2.2.1 - existence of compatible couples. Take any $f\in C^\infty_b(\mathbb{R};\mathbb{R})$ with $f'\leq 0$ and any $g\in C^\infty_b(\mathbb{R};\mathbb{R})$. Select any function ϕ_0 with

$$\phi_0\in C^\infty(\mathbb{R})\,, \qquad \phi_0'\in C^\infty_b(\mathbb{R})\,, \qquad \phi_0'(x_2)\,\geq\,2\,c\,>\,0\,, \quad \forall\,x_2\in\mathbb{R}\,.$$

Solve the quasilinear equation (2.20) where x_1 is interpreted as a time variable and the Cauchy data is $\phi(0,x_2) = \phi_0(x_2)$. It furnishes a solution of (2.20) which is defined on a domain of the form

$$\breve{\omega} \, = \, \{ \, x = {}^t(x_1,x_2) \in \mathbb{R}^2 \, ; \, |x_1| < \eta \, \} \, , \qquad \eta > 0 \, .$$

If necessary, restrict η to be sure that (2.14) is verified. Take any s^* and any $\bar{H}^1$. Extract $\bar{H}^2$ through (2.21). Then, piece together these ingredients to obtain a compatible couple (H,ϕ) which is defined on $\breve{\omega}$.

When f' is not constant, the construction of a compatible couple (H,ϕ) which is globally defined on $\mathbb{R}^2$ is not possible. This is due to (2.20). When $f'' \equiv 0$, the characteristics are straight lines which cross. $\diamond$

Introduce the family of lines

$$E(z) \, := \, \big\{ \, \lambda \, {}^t\big(-1, f(z)\big) \, ; \, \lambda \in \mathbb{R} \, \big\} \, \subset \, \mathbb{R}^2 \, , \qquad z \in \mathcal{I} \, .$$

The proof of Lemma 2.5 is based on a decomposition of the profile H in its oscillating part H^* and its mean value $\bar{H}$. Another point of view consists in looking at the vector valued function H in the basis $({}^t\nabla_x \phi, \nabla \phi^\perp)$. This changes the presentation of H.

Lemma 2.6. *Let $\phi \in C^\infty_b(\omega;\mathcal{I})$ satisfying (2.20). The profile H satisfies the relations (2.19) and (2.21) for some function $g \in C^\infty_b(\mathcal{I};\mathbb{R})$ if and only if there exists a scalar function $r \in C^\infty_b(\omega \times \mathbb{T};\mathbb{R})$ and two smooth vector field $H_\perp \in C^\infty_b(\mathcal{I};\mathbb{R}^2)$, and $H_\parallel \in C^\infty_b(\mathcal{I} \times \mathbb{R};\mathbb{R}^2)$ which are polarized according to*

$$H_\perp(z) \in \, E(z)^\perp \, , \qquad H_\parallel(z,y) \in \, E(z) \, , \qquad \forall \, (z,y) \in \mathcal{I} \times \mathbb{R} \tag{2.28}$$

such that the following decomposition holds

$$H(x,\theta) \, = \, H_\perp\big(\phi(x)\big) \, + \, H_\parallel\big(\phi(x), r(x,\theta)\big) \, , \qquad \forall \, (x,\theta) \in \omega \times \mathbb{T} \, . \tag{2.29}$$

Of course, when such a decomposition (2.29) exists, it is not unique: one can change the scalar function r and make the corresponding convenient change of the function $H_\parallel$ to get a new decomposition. In the sequel, functions like r will be called *underlying scalar profiles*.

<u>Proof</u> <u>of</u> <u>Lemma</u> <u>2.6</u>. Decompose H according to $H = H^c_\perp + H^c_\parallel$ with

$$\begin{aligned} H^c_\perp(x,\theta) \, &= \, |\nabla_x \phi(x)|^{-2} \, \big(\nabla_x \phi(x) \cdot H(x,\theta)\big) \, \nabla_x \phi(x) \, , \\ H^c_\parallel(x,\theta) \, &= \, |\nabla_x \phi(x)|^{-2} \, \big(\nabla_x \phi(x)^\perp \cdot H(x,\theta)\big) \, \nabla_x \phi(x)^\perp \, . \end{aligned}$$

Suppose first that (2.19) and (2.21) hold. Then

$$H^c_\perp(x,\theta) = H_\perp\big(\phi(x)\big) \, , \qquad H_\perp(z) := \big[1 + f(z)^2\big]^{-1} \, g(z) \, {}^t\big(f(z),1\big) \, .$$

On the other hand, with the choices

$$H_\parallel(z,y) := y\,{}^t\big(-1, f(z)\big)\,, \quad r(x,\theta) = \big[1+f(\phi)^2\big]^{-1}\,{}^t\big(-1, f(\phi)\big)\cdot H(x,\theta)\,,$$

we recover

$$H^c_\parallel(x,\theta) \,=\, H_\parallel\big(\phi(x), r(x,\theta)\big)\,.$$

With these definitions, we have (2.28) and (2.29). Conversely, suppose (2.28) and (2.29). Then

$$\nabla_x\phi(x)\cdot H^*(x,\theta) \,=\, \nabla_x\phi(x)\cdot\big[H_\parallel\big(\phi(x), r(x,\theta)\big)\big]^* \,=\, 0$$

and we obtain (2.21) with $g(z) = {}^t\big(f(z),1\big)\cdot H_\perp(z)$. □

2.3 Well prepared families for the Euler equation.

Our aim in this subsection is to identify the constraints to impose on ϕ and H in order to have

$$(2.30) \qquad h^\varepsilon \in \mathcal{V}_0^\tau(\omega)\,, \qquad \forall\,\varepsilon\in\,]0,1]\,.$$

We work with $\mathbf{f}_0 \equiv 0$. The other cases $\iota = 1$ and especially $\iota = -1$ are interesting but they will not be considered here.

Definition 2.2. *We say that the family $\{h^\varepsilon\}_\varepsilon$, where h^ε is defined as in (2.17) and is made of ingredients satisfying 2.a), $\cdots$, 2.d), is well prepared on ω for the incompressible Euler equation (1.1) if*

$$(2.31) \qquad I^1_{h^\varepsilon}(x) \,=\, \operatorname{div}_x h^\varepsilon(x) \,=\, 0\,, \qquad \forall\,(x,\varepsilon)\in\omega\times]0,1]\,,$$

$$(2.32) \qquad I^2_{h^\varepsilon}(x) \,=\, \det\big(D_x h^\varepsilon(x)\big) \,=\, 0\,, \qquad \forall\,(x,\varepsilon)\in\omega\times]0,1]\,.$$

In other words, a well prepared family $\{h^\varepsilon\}_\varepsilon$ for (1.1) is well prepared for (1.2) and is made of functions h^ε which are contained in $\mathcal{V}_0^\tau(\omega)$.

Suppose that $\{h^\varepsilon\}_\varepsilon$ is well prepared on ω for (1.1). According to Lemma 2.5, we have (2.17) with a phase ϕ and a principal profile H adjusted according to (2.19), (2.20) and (2.21). It is interesting to identify the other constraints satisfied by such ϕ and H. In fact, these supplementary conditions concern only the choice of g and of the function s^* defined in (2.23).

Lemma 2.7. *Select any functions $\phi\in C_b^\infty(\omega;\mathbb{R})$ and $f\in C_b^\infty(\mathcal{I};\mathbb{R})$ satisfying (2.20). Select any functions $K\in C_b^\infty(\mathcal{I}\times\mathbb{T};\mathbb{R})$ and $f_0\in C_b^\infty(\mathcal{I};\mathbb{R})$. There exist some open set $\breve\omega\subset\omega$ and a family $\{\breve h^\varepsilon\}_\varepsilon$ which is given by an asymptotic expansion like (2.17) with $J\geq 0$ and which is well prepared on $\breve\omega$ for (1.1).*

Moreover, the family $\{\breve{h}^\varepsilon\}_\varepsilon$ is associated with the phase ϕ and the profile

$$H(x,\theta) \,=\, K\big(\phi(x),\theta+\phi_0(x,\theta)\big) \begin{pmatrix} 1 \\ -f\circ\phi(x) \end{pmatrix} + \begin{pmatrix} 0 \\ g\circ\phi(x) \end{pmatrix} \tag{2.33}$$

where the function $g(z)$ giving rise to (2.21) is

$$g(z) \,:=\, K(0,0)\ f(0) \,+\, \int_0^z \bar{K}(y)\ f'(y)\ dy\,, \qquad z\in\mathcal{I} \tag{2.34}$$

and where ϕ_0 is subjected to the scalar quasilinear equation

$$\begin{aligned} \partial_1\phi_0 \,-\, f(\phi)\ \partial_2\phi_0 + \big(-f'(\phi)\ \phi_0 + f_0(\phi)\big)\ \partial_2\phi\ \partial_\theta\phi_0 & \\ + \big(-f'(\phi)\ \phi_0 + f_0(\phi)\big)\ \partial_2\phi \,=\, 0\,. & \end{aligned} \tag{2.35}$$

<u>Proof</u> <u>of</u> <u>Lemma</u> *2.7.* The restriction (2.32) means that we can find

$$\tilde{h}^\varepsilon \,=\, {}^t(\tilde{h}^{\varepsilon 1},\tilde{h}^{\varepsilon 2}) \in C^\infty_b(\mathcal{I};\mathbb{R}^2)\,, \qquad \varphi^\varepsilon \in C^\infty(\omega;\mathcal{I})\,, \qquad \varepsilon\in]0,1]$$

such that

$$\breve{h}^\varepsilon(x) \,=\, {}^t\big(\breve{h}^{\varepsilon 1}(x),\breve{h}^{\varepsilon 2}(x)\big) \,=\, \tilde{h}^\varepsilon\big(\varphi^\varepsilon(x)\big)\,, \qquad \forall\,(x,\varepsilon)\in\omega\times]0,1]\,. \tag{2.36}$$

Choose any sequence

$$K_j \in C^\infty(\mathcal{I}\times\mathbb{T};\mathbb{R})\,, \qquad j\in\{0,\cdots,J-1\}\,.$$

Define

$$\tilde{h}^{\varepsilon 1}(z) = K^\varepsilon\big(z,\tfrac{z}{\varepsilon}\big) := K\big(z,\tfrac{z}{\varepsilon}\big) + \varepsilon\ \textstyle\sum_{j=0}^{J-1}\ \varepsilon^j\ K_j\big(z,\tfrac{z}{\varepsilon}\big)\,, \qquad (z,\varepsilon)\in\mathcal{I}\times\,]0,1]\,,$$

$$\tilde{h}^{\varepsilon 2}(z) = \textstyle\int_0^z \big(-f(y)+\varepsilon\, f^\varepsilon(y)\big)\ (\tilde{h}^{\varepsilon 1})'(y)\ dy\,, \qquad (z,\varepsilon)\in\mathcal{I}\times]0,1]\,,$$

where

$$f^\varepsilon(z) \,=\, \textstyle\sum_{j=0}^{J-1}\ \varepsilon^j\ f_j(z)\,, \qquad f_j\in C^\infty_b(\mathbb{R};\mathbb{R})\,, \qquad j\in\{0,\cdots,J-1\}\,.$$

Two integrations by parts lead to

$$\begin{aligned} \tilde{h}^{\varepsilon 2}(z) \,=\, & \big(-f(z)+\varepsilon\, f^\varepsilon(z)\big)\ \tilde{h}^{\varepsilon 1}(z) \,-\, \big(-f(0)+\varepsilon\, f^\varepsilon(0)\big)\ \tilde{h}^{\varepsilon 1}(0) \\ & -\, \textstyle\int_0^z \big(-f'(y)+\varepsilon\,(f^\varepsilon)'(y)\big)\ \bar{K}^\varepsilon(y)\ dy \\ & -\varepsilon\ \big(-f'(z)+\varepsilon\,(f^\varepsilon)'(z)\big)\ (\partial_\theta^{-1}K^{\varepsilon *})\big(z,\tfrac{z}{\varepsilon}\big) \\ & +\varepsilon\ \big(-f'(0)+\varepsilon\,(f^\varepsilon)'(0)\big)\ (\partial_\theta^{-1}K^{\varepsilon *})(0,0) \\ & +\varepsilon\ \textstyle\int_0^z \big(-f''(y)+\varepsilon\,(f^\varepsilon)''(y)\big)\ (\partial_\theta^{-1}K^{\varepsilon *})\big(y,\tfrac{y}{\varepsilon}\big)\ dy \\ & +\varepsilon\ \textstyle\int_0^z \big(-f'(y)+\varepsilon\,(f^\varepsilon)'(y)\big)\ (\partial_y\partial_\theta^{-1}K^{\varepsilon *})\big(y,\tfrac{y}{\varepsilon}\big)\ dy\,. \end{aligned}$$

The procedure can be repeated since for instance

$$(\partial_\theta^{-1}K^{\varepsilon *})\big(y,\tfrac{y}{\varepsilon}\big) \,=\, \varepsilon\,\partial_y\big[\,(\partial_\theta^{-1}\partial_\theta^{-1}K^{\varepsilon *})\big(y,\tfrac{y}{\varepsilon}\big)\,\big] - \varepsilon\,(\partial_y\,\partial_\theta^{-1}\partial_\theta^{-1}K^{\varepsilon *})\big(y,\tfrac{y}{\varepsilon}\big)\,.$$

It furnishes

$$\tilde h^{\varepsilon 2}(z) \,=\, \tilde H^2\big(z,\tfrac{z}{\varepsilon}\big) \,+\, \varepsilon \,\sum_{j=0}^{J-1} \varepsilon^j \, \tilde H^2_j\big(z,\tfrac{z}{\varepsilon}\big) \,+\, O(\varepsilon^{J+1})$$

with in particular

$$\tilde H^2(z,\theta) \,:=\, -\, f(z) \; K(z,\theta) \,+\, f(0)\; K(0,0) \,+\, \int_0^z \bar K(y) \; f'(y) \; dy \,, \tag{2.37}$$

$$\begin{aligned} \tilde H^2_0(z,\theta) \,:=\, & -f(z)\; K_0(z,\theta) \,+\, f_0(z)\; K(z,\theta) \,+\, f(0)\; K_0(0,0) \\ & -\, f_0(0)\; K(0,0) \,+\, \int_0^z \, f'(y)\; \bar K_0(y) \; dy \,-\, \int_0^z \, f'_0(y)\; \bar K(y)\; dy \\ & +\, f'(z)\; (\partial_\theta^{-1} K^*)(z,\theta) \,-\, f'(0)\; (\partial_\theta^{-1} K^*)(0,0)\,. \end{aligned} \tag{2.38}$$

The expression $\tilde h^\varepsilon$ is adjusted so that (2.31) becomes the consequence of

$$\partial_1 \varphi^\varepsilon \,+\, \big(-f(\varphi^\varepsilon) + \varepsilon\, f^\varepsilon(\varphi^\varepsilon)\big)\; \partial_2 \varphi^\varepsilon \,=\, 0\,. \tag{2.39}$$

Seek solutions φ^ε of (2.39) in the form

$$\varphi^\varepsilon(x) \,=\, \phi(x) \,+\, \varepsilon\; \phi^\varepsilon\big(x,\phi(x)/\varepsilon\big)\,, \qquad \phi^\varepsilon \in C^\infty_b(\breve\omega \times \mathbb{T};\mathbb{R}) \tag{2.40}$$

where $\phi(x)$ is subjected to (2.20) and $\phi^\varepsilon(x,\theta)$ can be expanded according to

$$\phi^\varepsilon(x,\theta) \,=\, \sum_{j=0}^{J+1} \varepsilon^j \; \phi_j(x,\theta) \,+\, \varepsilon^{J+2}\; r\phi^\varepsilon(x,\theta)\,.$$

The expression ϕ^ε must satisfy

$$\partial_1 \phi^\varepsilon \,+\, a(\varepsilon,t,x,\phi^\varepsilon)\; \partial_2 \phi^\varepsilon \,+\, b(\varepsilon,t,x,\phi^\varepsilon)\; \partial_\theta \phi^\varepsilon \,+\, b(\varepsilon,t,x,\phi^\varepsilon) \,=\, 0 \tag{2.41}$$

with

$$a(\varepsilon,t,x,z) \,:=\, -\, f(\phi + \varepsilon\, z) \,+\, \varepsilon\; f^\varepsilon(\phi + \varepsilon\, z)\,,$$

$$b(\varepsilon,t,x,z) \,:=\, \big[-z\; \int_0^1 \, f'(\phi + \varepsilon\, s\, z)\; ds \,+\, f^\varepsilon(\phi + \varepsilon\, z)\big]\; \partial_2 \phi\,.$$

Look at (2.41) as an evolution equation in the variable x_1. Remark that $\mathcal{J} := \omega(0) \subset \mathbb{R}$ is an open interval containing 0. Choose functions

$$\phi_{0j} \,\in\, C^\infty_b(\mathcal{J}\times\mathbb{T};\mathbb{R})\,, \qquad r\phi^\varepsilon_0 \,\in\, C^\infty_b(\mathcal{J}\times\mathbb{T};\mathbb{R})\,, \qquad j \in \{0,\cdots,J+1\}$$

where the family $\{r\phi^\varepsilon_0\}_\varepsilon$ is such that

$$\sup\,\big\{\, \|\, r\phi^\varepsilon_0 \,\|_{L^\infty(\mathcal{J}\times\mathbb{T})} \,+\varepsilon^{J-2} \;\; \|\, D_{x_2,\theta}\, r\phi^\varepsilon_0 \,\|_{L^\infty(\mathcal{J}\times\mathbb{T})}\,;\; \varepsilon \in]0,1] \,\big\} \,<\, \infty\,.$$

Complete (2.41) with the initial data

$$\phi^\varepsilon(0,x_2,\theta) \,=\, \phi^\varepsilon_0(x_2,\theta)\,, \qquad \phi^\varepsilon_0 \,=\, \sum_{j=0}^{J+1} \varepsilon^j \; \phi_{0j} \,+\, \varepsilon^{J+2}\; r\phi^\varepsilon_0\,. \tag{2.42}$$

Under these conditions, it is possible to find some open set $\breve\omega \subset \omega \subset \mathbb{R}^2$ with $\breve\omega(0) = \mathcal{J}$ and, for all $\varepsilon \in]0,1]$, a solution ϕ^ε of (2.41)-(2.42) on $\breve\omega$.

The term ϕ_0 is subjected to (2.35) and the remainder $r\phi^\varepsilon$ is controlled by

$$\sup \left\{ \| \, r\phi^\varepsilon \, \|_{L^\infty(\breve{\omega}\times\mathbb{T})} \, + \varepsilon^{J-2} \ \| \, D_{x,\theta} \, r\phi^\varepsilon \, \|_{L^\infty(\breve{\omega}\times\mathbb{T})} \, ; \ \varepsilon \in]0,1] \right\} \, < \, \infty \, .$$

By way of formula (2.40), this furnishes a solution φ^ε of (2.39). Plug φ^ε in $\tilde{h}^\varepsilon$ as in (2.36) to get with $K_{-1} \equiv K$ and $\tilde{H}^2_{-1} \equiv \tilde{H}^2$

$$\breve{h}^{\varepsilon 1}(x) \, = \, \textstyle\sum_{j=-1}^{J-1} \, \varepsilon^{j+1} \, K_j\big(\phi(x) + \varepsilon \, \phi^\varepsilon(x), \phi(x)/\varepsilon + \phi^\varepsilon(x)\big) \, ,$$

$$\breve{h}^{\varepsilon 2}(x) \, = \, \textstyle\sum_{j=-1}^{J-1} \, \varepsilon^{j+1} \, \tilde{H}^2_j\big(\phi(x) + \varepsilon \, \phi^\varepsilon(x), \phi(x)/\varepsilon + \phi^\varepsilon(x)\big) \, + \, O\big(\varepsilon^{J+1}\big) \, .$$

Use a Taylor formula to recover (2.17) with ingredients satisfying 2.a), $\cdots$, 2.d). The relation (2.37) leads to (2.33) with g as indicated. Obviously, the family $\{\breve{h}^\varepsilon\}_\varepsilon$ is well prepared on $\breve{\omega}$ for (1.1). $\square$

We can also deduce from the preceding construction the following more refined informations.

Lemma 2.8. *Select any functions $K_0 \in C^\infty_b(\mathcal{I} \times \mathbb{T}; \mathbb{R})$ and $f_1 \in C^\infty_b(\mathcal{I}; \mathbb{R})$. In the framework of Lemma 2.7 applied with $J \geq 1$, the second profile H_0 can be put in the form*

$$H_0(x,\theta) \, = \, \breve{H}_0\big(x, \phi(x), \theta + \phi_0(x,\theta)\big) \, , \qquad \breve{H}_0 \in C^\infty_b(\breve{\omega} \times \mathbb{R} \times \mathbb{T}; \mathbb{R}^2) \, .$$

Above, the function $\breve{H}_0 = {}^t(\breve{H}^1_0, \breve{H}^2_0)$ can be adjusted so that

$$\breve{H}^1_0(x,z,\theta) \, = \, \phi_0(x,\theta) \ (\partial_z K)(z,\theta) \, + \, \phi_1(x,\theta) \ (\partial_\theta K)(z,\theta) \, + \, K_0(z,\theta) \, ,$$

$$\breve{H}^2_0(x,z,\theta) \, = \, \phi_0(x,\theta) \ (\partial_z \tilde{H}^2)(z,\theta) \, + \, \phi_1(x,\theta) \ (\partial_\theta \tilde{H}^2)(z,\theta) \, + \, \tilde{H}^2_0(z,\theta) \, ,$$

with $\tilde{H}^2$ and $\tilde{H}^2_0$ defined according to (2.37) and (2.38), and with ϕ_1 subjected to the scalar quasilinear equation

$$\begin{aligned} &\partial_1\phi_1 \, - \, f(\phi) \ \partial_2\phi_1 \, + \, \partial_2\phi \ \big(- f'(\phi) \ \phi_0 + f_0(\phi)\big) \ \partial_\theta\phi_1 \\ &\quad - \partial_2\phi \ (1 + \partial_\theta\phi_0) \ f'(\phi) \ \phi_1 \, + \, \partial_2\phi_0 \ \big(- f'(\phi) \ \phi_0 + f_0(\phi)\big) \\ &\quad + \partial_2\phi \ (1 + \partial_\theta\phi_0) \ \big(- \tfrac{1}{2} \ f''(\phi) \ \phi_0^2 + f_0'(\phi) \ \phi_0 + f_1(\phi)\big) \, = \, 0 \, . \end{aligned} \tag{2.43}$$

Remark 2.3.2 - more general constructions. The preceding description of families which are well prepared for (1.1) is not exhaustive. For instance, for all $j \in \{0, \cdots, J-1\}$, the function $f_j(z)$ can be replaced by $f_j\big(z, \frac{z}{\varepsilon}\big)$ which yields more complicated formulae. $\diamond$

3 Simple waves.

A simple wave is a solution of (1.1) or (1.2) having the form (1.11) where the profile $\mathbf{H}$ and the phase Φ do not depend on ε. The functions $\mathbf{H}$ and Φ are chosen smooth on some domain of determinacy $\Omega \subset \mathbb{R} \times \mathbb{R}^2$, say

$$\mathbf{H} \in C^\infty_b(\Omega \times \mathbb{T}; \mathbb{R}^2) \, , \qquad \Phi \in C^\infty(\Omega; \mathbb{R}) \, , \qquad \omega = (\{0\} \times \mathbb{R}^2) \cap \Omega \, .$$

3.1 The case of Burger equations.

The aim of this subsection 3.1 is to construct *all* simple waves which are associated with (1.2). Suppose that a family $\{\tilde{\mathbf{u}}^\varepsilon\}_{\varepsilon\in]0,1]}$ is made of C^1 solutions on Ω to the Burger equation (1.2) and is given by a formula like (1.11). Then, the corresponding initial data is

$$\mathbf{H}\big(0,x,\Phi(0,x)/\varepsilon\big) \,=\, H\big(x,\phi(x)/\varepsilon\big)$$

with

$$H(x,\theta) \,=\, {}^t(H^1,H^2)(x,\theta) \,:=\, \mathbf{H}(0,x,\theta)\,, \qquad \phi(x) \,:=\, \Phi(0,x)\,.$$

To avoid inside Ω the crossing of characteristics, necessarily the couple (H,ϕ) must be compatible. It means that the expressions H and ϕ are adjusted as in Lemma 2.5, with ingredients f, g and s^* yielding (2.20), (2.21) and (2.23). Now, there is a natural way to associate with such (H,ϕ) a simple wave. Define

$$s(x,\theta) \,:=\, -\,\bar{H}^1(x)\,/\,\partial_2\phi(x) \,+\, s^*(x,\theta) \in C^1(\omega\times\mathbb{T};\mathbb{R})\,.$$

Introduce the eiconal equation

$$\partial_t\Phi \,+\, g(\Phi)\;\partial_2\Phi \,=\, 0 \tag{3.1}$$

and the quasi-linear equation

$$\begin{aligned}\partial_t\mathbf{s} \,+\, g(\Phi)\;\partial_2\mathbf{s} \,+\, \mathbf{s}\;(-\partial_2\Phi\;\partial_1+\partial_1\Phi\;\partial_2)\,\mathbf{s}&\\ =\,\partial_2\Phi\;\mathbf{s}\;\big(g'(\Phi)+f'(\Phi)\;\partial_2\Phi\;\mathbf{s}\big)\,-\,|\nabla_x\Phi|^{-2}\;{}^t\nabla_x\Phi^\perp\cdot\mathbf{f}\,.&\end{aligned} \tag{3.2}$$

Complete these equations with the initial data

$$\Phi(0,x)=\phi(x)\,,\qquad \mathbf{s}(0,x,\theta)=s(x,\theta)\,. \tag{3.3}$$

Lemma 3.1. *Select any compatible couple (H,ϕ). There is a domain of determinacy Ω^τ of the form*

$$\Omega^\tau \,:=\, \big\{\,(t,x)\in[0,\tau[\times\mathbb{R}^2\,;\;B(x,c^\tau\;t[\subset\omega\,\big\}\,,\qquad \tau\in]0,1]$$

such that the Cauchy problems (3.1)-(3.3) and (3.2)-(3.3) have solutions Φ and $\mathbf{s}$ respectively on Ω^τ and $\Omega^\tau\times\mathbb{T}$. With these ingredients, build the large amplitude wave

$$\begin{aligned}\tilde{\mathbf{u}}^\varepsilon(t,x)\,&=\,\mathbf{H}\big(t,x,\Phi(t,x)/\varepsilon\big)\\ &:=\,\mathbf{s}\big(t,x,\Phi(t,x)/\varepsilon\big)\,\begin{pmatrix}-\partial_2\Phi(t,x)\\ \partial_1\Phi(t,x)\end{pmatrix}+\begin{pmatrix}0\\ g\big(\Phi(t,x)\big)\end{pmatrix}.\end{aligned} \tag{3.4}$$

At the time $t = 0$, one has

$$\tilde{\mathbf{u}}^\varepsilon(0,x) \,=\, \tilde h^\varepsilon(x) \,:=\, H\big(x,\phi(x)/\varepsilon\big)\,.$$

Assume that the source term $\mathbf{f}(t,x)$ is adjusted so that $\nabla_x\Phi\cdot\mathbf{f} \equiv 0$. Then, the expression $\tilde{\mathbf{u}}^\varepsilon$ is a simple wave which is a solution on Ω^τ of (1.2). Moreover, the relations (2.20) and (2.21) are conserved during the evolution.

Proof of Lemma 3.1. Start by checking the initial data. Use (2.20), (2.21) and (2.23) to obtain

$$\begin{aligned}\mathbf{H}(0,x,\theta) \,&=\, \mathbf{s}(0,x,\theta)\ \nabla_x\phi(x)^\perp \,+\, {}^t\big(0\,,\, g\circ\phi(x)\big)\\ &=\bar H(x) \,+\, s^*(x,\theta)\ \nabla_x\phi(x)^\perp \,=\, H(x,\theta)\,.\end{aligned}$$

The definition (3.4) clearly implies that

$$\mathbf{H}^*(t,x,\theta)\cdot\nabla_x\Phi(t,x) \,=\, 0\,,\qquad \forall\,(t,x,\theta)\in\Omega^\tau\times\mathbb{T}\,.$$

Plug (3.4) at the level of (1.2). Say that the expressions with ε^{-1} and ε^0 in factor are separately equal to zero. Then, collect all the preceding informations to get the system (1.12) which is over-determined. For general choices of (H,ϕ), the system (1.12) has no solution.

In fact, the matter is to show that the Cauchy problem (1.12) is (locally in time) well-posed once the couple (H,ϕ) is compatible with (1.2). Lemma 3.1 proposes to solve (1.12) by using the formula

$$\mathbf{H}(t,x,\theta) \,=\, \mathbf{s}(t,x,\theta)\ \begin{pmatrix} -\partial_2\Phi(t,x)\\ \partial_1\Phi(t,x)\end{pmatrix} + \begin{pmatrix} 0\\ g\big(\Phi(t,x)\big)\end{pmatrix}. \tag{3.5}$$

This formulation (3.5) contains obviously the polarization condition on $\mathbf{H}^*$. It is adjusted so that the second equation of (1.12) (on Φ) gives rise to (3.1). From (3.1), it is also possible to extract

$$\big[\,\partial_t \,+\, g(\Phi)\ \partial_2 \,+\, \partial_2\Phi\ g'(\Phi)\,\big]\,\big(\partial_1\Phi \,-\, f(\Phi)\ \partial_2\Phi\big) \,=\, 0\,.$$

It follows that the relation (2.20) and hence the condition (2.21) are propagated during the evolution. In other words

$$\partial_1\Phi(t,x) \,=\, f\big(\Phi(t,x)\big)\ \partial_2\Phi(t,x)\,,\qquad \forall\,(t,x)\in\Omega^\tau\,, \tag{3.6}$$

$$\bar{\mathbf{H}}^2(t,x) + f\big(\Phi(t,x)\big)\ \bar{\mathbf{H}}^1(t,x) \,=\, g\big(\Phi(t,x)\big)\,,\qquad \forall\,(t,x)\in\Omega^\tau\,. \tag{3.7}$$

Consider now the first equation in (1.12). Because of (3.2), it reduces to

$$\begin{aligned}&\partial_t\nabla_x\Phi^\perp \,+\, g(\Phi)\ \partial_2\nabla_x\Phi^\perp \,+\, \partial_2\Phi\ g'(\Phi)\ \nabla_x\Phi^\perp\\ &\qquad +\mathbf{s}\,\big[\,(\nabla_x\Phi^\perp\cdot\nabla_x)\,\nabla_x\Phi^\perp \,+\, f'(\Phi)\ (\partial_2\Phi)^2\ \nabla_x\Phi^\perp\,\big] \,=\, 0\,.\end{aligned} \tag{3.8}$$

On the one hand, the equation (3.6) guarantees that

$$(-\partial_2\Phi\ \partial_1+\partial_1\Phi\ \partial_2)\,\nabla_x\Phi^\perp\ =\ -\,f'(\Phi)\ (\partial_2\Phi)^2\ \nabla_x\Phi^\perp\,.$$

On the other hand, the derivation of (3.1) furnishes

$$\partial_t\nabla_x\Phi^\perp\ +\ g(\Phi)\ \partial_2\nabla_x\Phi^\perp\ +\ \partial_2\Phi\ g'(\Phi)\ \nabla_x\Phi^\perp\ =\ 0\,.$$

Thus, the identity (3.8) is verified. □

There is another way to consider $\mathbf{H}$. This point of view consists in appealing to the framework of Lemma 2.6.

Lemma 3.2. *The profile* $\mathbf{H}$ *can be written*

$$\mathbf{H}(t,x,\theta)\ =\ \tilde H\big(\Phi(t,x),\mathbf{r}(t,x,\theta)\big)\,,\qquad \tilde H(z,y)\ =\ H_\perp(z)\ +\ H_\parallel(z,y)$$

where the phase Φ *is given by*

$$\partial_t\Phi\ +\ \big(H_\perp(\Phi)\cdot\nabla_x\big)\Phi\ =\ 0\,,\qquad \Phi(0,x)\ =\ \phi(x) \tag{3.9}$$

whereas the function $\mathbf{r}$ *is obtained by solving the scalar quasilinear equation*

$$\left\{\begin{array}{l}\partial_t\mathbf{r}\ +\ \big(\tilde H(\Phi,\mathbf{r})\cdot\nabla_x\big)\mathbf{r}\ +\ \big(1+f(\Phi)^2\big)^{-1}\ {}^t\big(-1,f(\Phi)\big)\cdot\mathbf{f}\ =\ 0\,,\\ \mathbf{r}(0,x,\theta)\ =\ r(x,\theta)\,.\end{array}\right. \tag{3.10}$$

Remark 3.1.1 - about $\mathbf{r}$. Thus, to solve (1.12), it suffices first to consider the eiconal equation (3.9) and then to look at the transport equation (3.10). Again, the function $\mathbf{r}$ is called an *underlying scalar profile*. Taking into account the definitions of $H_\perp$ and $H_\parallel$, we get the decomposition

$$\mathbf{H}(t,x,\theta)=\frac{g(\Phi)}{1+f(\Phi)^2}\left(\begin{array}{c}f(\Phi)\\1\end{array}\right)+\mathbf{r}(t,x,\theta)\left(\begin{array}{c}-1\\f(\Phi)\end{array}\right)\,,\qquad \Phi=\Phi(t,x)\,.$$

◇

Proof of Lemma 3.2. In view of Lemma 2.6, we have

$$\mathbf{H}(0,x,\theta)\ =\ H(x,\theta)\ =\ \tilde H\big(\phi(x),r(x,\theta)\big)\,.$$

The solution of (1.12), once it exists, is unique. Therefore, it suffices to show that the expressions Φ and $\mathbf{H}$ which can be extracted from (3.9) and (3.10) satisfy the system (1.12). The condition (3.9) implies that

$$\big[\,\partial_t\ +\ H_\perp(\Phi)\cdot\nabla_x\ +\ H_\perp'(\Phi)\cdot\nabla_x\Phi\,\big]\,\big(\partial_1\Phi\ -\ f(\Phi)\ \partial_2\Phi\big)\ =\ 0\,.$$

One has (3.6) which means that $\nabla_x\Phi\perp E(\Phi)$. Using (2.28), it gives rise to

$$\bar{\mathbf{H}}\cdot\nabla_x\Phi\ =\ H_\perp(\Phi)\cdot\nabla_x\Phi\,,\qquad \nabla_x\Phi\cdot\mathbf{H}^*\ =\ \nabla_x\Phi\cdot H_\parallel(\Phi,\mathbf{r})^*\ =\ 0\,.$$

Thus, (3.9) is the same as the second equation of (1.12) whereas the third equation of (1.12) is verified. It remains to compute

$$\partial_t \mathbf{H} \,+\, (\mathbf{H}\cdot\nabla_x)\mathbf{H} \,+\, \mathbf{f} \;=\; \big[\,\partial_t \mathbf{r} \,+\, (\mathbf{H}\cdot\nabla_x)\mathbf{r}\,\big]\; \partial_y H_{\|}(\Phi,\mathbf{r}) \,+\, \mathbf{f}$$

which is equal to zero because of (3.10). □

Remark 3.1.2 - weak convergence. Look at the special choices

$$\mathbf{s}(t,x,\theta) \;=\; x_1\;\cos(2\,\pi\,\theta)\,/\,\big(1+t\;\cos(2\,\pi\,\theta)\big)\,, \qquad \Phi(t,x) \;=\; -x_2\,,$$

which satisfy all the required conditions (with $f\equiv 0$, $g\equiv 0$ and $\mathbf{f}\equiv 0$). The weak limit $\bar{\mathbf{u}}$ of the corresponding family $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$ is

$$\bar{\mathbf{u}}(t,x) \;=\; x_1\;\int_0^1 \big[\cos(2\,\pi\,\theta)\,/\,\big(1+t\;\cos(2\,\pi\,\theta)\big)\big]\;d\theta\;\;{}^t(1,0)\,.$$

Observe that $\bar{\mathbf{u}}$ is not a solution of (1.2) since

$$\big(\partial_t \bar{\mathbf{u}}_1 + (\bar{\mathbf{u}}\cdot\nabla_x)\bar{\mathbf{u}}_1\big)(0,x) \;=\; -\,x_1\;\int_0^1 \big(\cos(2\,\pi\,\theta)\big)^2\;d\theta \;\not\equiv\; 0\,.$$

It means that (1.2) is not closed for the weak topology of L^2. Now, this is a very natural expectation since (1.2) has no conservative form. ◇

Remark 3.1.3 - special diffeomorphisms. Select again a couple (H,ϕ) which is compatible with (1.2). The application

$$\Xi^\varepsilon_t\;:\;x\longmapsto\;x\,+\,t\;H\big(x,\phi(x)/\varepsilon\big)$$

is a local diffeomorphism whose inverse can be made explicit. It is

$$(\Xi^\varepsilon_t)^{-1}\;:\;x\longmapsto\;x\,-\,t\;\mathbf{H}\big(t,x,\Phi(t,x)/\varepsilon\big)$$

where $\mathbf{H}$ and Φ are obtained by solving the nonlinear system (1.12). ◇

Remark 3.1.4 - caustics of the first type. There is no constraint (except the regularity) on the choice of g. When the function g is non increasing the solution Φ of (3.1) develops shocks which correspond to the formation of caustics for the Burger equation. ◇

3.2 The case of incompressible equations.

The study of the over-determined system (1.16) is more delicate than the one of (1.12). We restrict here our attention to the case $\mathbf{p}\equiv\mathbf{p}_\iota$. In this particular context, the construction of simple waves which are incompressible needs to restrict the choice of the ingredients f, g and $\mathbf{r}$.

Lemma 3.3. *Among the simple waves which are solutions to (1.2) where $\mathbf{f}=\nabla_x\mathbf{p}_\iota$ for some $\iota\in\{-1,0,1\}$, that is which are given by Lemma 3.1, those who satisfy also (1.16) are plane waves.*

More precisely, they take the form

$$\check{\mathbf{u}}^\varepsilon(t,x) \,=\, \Big[k\Big(z,\frac{\psi(z)}{\varepsilon}\Big)\ \psi'(z)\Big]_{|\, z=a\,t+b\,x_1+x_2}\ \begin{pmatrix} -1 \\ b \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \tag{3.11}$$

where $k \in C_b^\infty(\mathbb{R}\times\mathbb{T};\mathbb{R})$, $\psi \in C^\infty(\mathbb{R};\mathbb{R})$ *and* $(a,b)\in\mathbb{R}^2$.

Proof of Lemma 3.3. Select a general simple wave $\tilde{\mathbf{u}}^\varepsilon$ as in (3.4), involving ingredients Φ, $\mathbf{s}$ and $\mathbf{H}$ as in (3.1), (3.2) and (3.5). In fact, the matter is to solve (1.12) where $\mathbf{f}\equiv\mathbf{f}_\iota$ and where the condition $\operatorname{div}_x\mathbf{H}\equiv 0$ is added. This amounts to the same thing as looking at the scalar quasilinear equation (3.2) completed with the constraints

$$\nabla_x\Phi^\perp\cdot\nabla_x\mathbf{s}^* \,=\, 0\,,\qquad \nabla_x\Phi^\perp\cdot\nabla_x\bar{\mathbf{s}} \,+\, g'(\Phi)\ \partial_2\Phi \,=\, 0\,. \tag{3.12}$$

The first condition means that $\mathbf{s}^*(t,\cdot)$ is a function of $\Phi(t,\cdot)$. The second condition, when imposed at time $t=0$, must be preserved during the evolution induced by (3.2). The initial data ϕ and s must be adjusted to this end. However, it is delicate to identify at the level of (3.2) the constraints to impose on ϕ and s. The difficulties are due to the interplay between the oscillating part $\mathbf{s}^*$ and the mean value $\bar{\mathbf{s}}$.

Below, we adopt another point of view. Our argument consists in appealing to the criterion of the paragraph 2.1.3. The expression

$$\tilde h^\varepsilon(x) \,=\, \tilde{\mathbf{u}}^\varepsilon(0,x) \,=\, \mathbf{H}\big(0,x,\Phi(0,x)/\varepsilon\big) \,=\, H\big(x,\phi(x)/\varepsilon\big)$$

must satisfy

$$\operatorname{div}_x\tilde h^\varepsilon(x) \,=\, 0\,,\qquad \det D_x\tilde h^\varepsilon(x) \,=\, \iota\,,\qquad \forall\, x\in\omega\,. \tag{3.13}$$

The divergence free condition yields

$$\operatorname{div}_x\tilde h^\varepsilon(x) \,=\, \nabla_x s\big(x,\phi(x)/\varepsilon\big)\cdot\nabla_x\phi(x)^\perp \,+\, g'\big(\phi(x)\big)\ \partial_2\phi(x) \,=\, 0\,.$$

Pass to the weak limit ($\varepsilon\longrightarrow 0$) to find

$$\nabla_x s(x,\theta)\cdot\nabla_x\phi(x)^\perp \,+\, g'\big(\phi(x)\big)\ \partial_2\phi(x) \,=\, 0\,,\qquad \forall\,(x,\theta)\in\omega\times\mathbb{T}\,.$$

Separate the mean value

$$-\,\partial_2\phi(x)\ \partial_1\bar s(x) \,+\, \partial_1\phi(x)\ \partial_2\bar s(x) \,+\, g'\big(\phi(x)\big)\ \partial_2\phi(x) \,=\, 0 \tag{3.14}$$

from the oscillating part

$$\nabla_x s^*(x,\theta)\cdot\nabla_x\phi(x)^\perp \,=\, 0\,. \tag{3.15}$$

As expected, the conditions (3.14) and (3.15) are the same as (3.12) written at time $t=0$. In particular, there is $k^* \in C_b^\infty(\mathcal{I}\times\mathbb{T};\mathbb{R})$ such that

$$s^*(x,\theta) \,=\, k^*\big(\phi(x),\theta\big)\,, \qquad k^* \not\equiv 0\,, \qquad \forall\,(x,\theta)\in\omega\times\mathbb{T}\,. \tag{3.16}$$

For $j=1$ or for $j=2$, one has

$$\begin{aligned}\partial_j \tilde h^\varepsilon(x) \,=\, & \,\partial_j\left[\partial_2\phi(x)\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)\right]\ {}^t\big(-1,f(\phi(x))\big)\\ & +\partial_j\phi(x)\ \left[\partial_2\phi(x)\ f'\big(\phi(x)\big)\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)+g'\big(\phi(x)\big)\right]\ {}^t(0,1)\,.\end{aligned}$$

Mark the coefficients

$$\begin{aligned}a^\varepsilon(x) \,:=\, & \,\partial_2\phi(x)\ f'\big(\phi(x)\big)\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)\,+\,g'\big(\phi(x)\big)\\ =\, & \,\bar a(x)\,+\,\bar m_a(x)\ k^*\big(\phi(x),\tfrac{\phi(x)}{\varepsilon}\big)\,.\\ b^\varepsilon(x) \,:=\, & \,\partial_1\left[\partial_2\phi(x)\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)\right]\,-\,f\big(\phi(x)\big)\ \partial_2\left[\partial_2\phi(x)\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)\right]\,=\,0\,.\\ =\, & \,\partial_2\phi(x)\ \partial_1\bar s(x)-\partial_1\phi(x)\ \partial_2\bar s(x)+f'\big(\phi(x)\big)\ \big(\partial_2\phi(x)\big)^2\ s\big(x,\tfrac{\phi(x)}{\varepsilon}\big)\\ =\, & \,\bar b(x)\,+\,\bar m_b(x)\ k^*\big(\phi(x),\tfrac{\phi(x)}{\varepsilon}\big)\,.\end{aligned}$$

Now, examine the second condition in (3.13) which is equivalent to

$$\partial_2\phi(x)\ a^\varepsilon(x)\ b^\varepsilon(x)\,+\,\iota\,=\,0\,, \qquad \forall\,\varepsilon\in\,]0,1]\,.$$

Pass to the weak limit ($\varepsilon\longrightarrow 0$) to find

$$\begin{aligned}&\bar a(x)\ \bar b(x)\,+\,\iota/\partial_2\phi(x)\,+\,\big(\bar m_a(x)+\bar m_b(x)\big)\ k^*\big(\phi(x),\theta\big)\\ &\qquad +\bar m_a(x)\ \bar m_b(x)\ k^*\big(\phi(x),\theta\big)^2\,=\,0\,.\end{aligned}$$

Since this must be true for all $(x,\theta)\in\omega\times\mathbb{T}$, this is the same as

$$\bar a(x)\ \bar b(x)=0\,, \qquad \bar m_a(x)+\bar m_b(x)=0\,, \qquad \bar m_a(x)\ \bar m_b(x)=0\,.$$

In particular

$$\bar m_a(x)\ \bar m_b(x)\,=\,\big(\partial_2\phi(x)\big)^3\ f'\big(\phi(x)\big)^2\,=\,0\,.$$

Necessarily f is a constant, say $f=b\in\mathbb{R}$. In view of (2.20), one has

$$\exists\,\psi\in C^\infty(\mathbb{R};\mathbb{R})\,; \qquad \pm\psi'\,>\,0\,, \qquad \phi(x)\,=\,\psi(b\,x_1+x_2)\,.$$

Now, remember that the construction of Lemma 3.1 requires a condition on $\mathbf{f}$. This condition concerns here $\mathbf{f}_\iota$ and, at time $t=0$, it is

$$\nabla_x\Phi(0,x)\cdot\mathbf{f}_\iota(0,x)\,=\,\iota\ (b\,x_1+x_2)\ \psi'(b\,x_1+x_2)\,=\,0\,, \qquad \forall\,x\in\omega\,.$$

Since $\psi'\not\equiv 0$, this is possible only if $\iota=0$. Therefore, the cases $\iota=-1$ and $\iota=1$ are excluded. From now on, take $\iota=0$.

Use (3.14) to extract

$$\bar{a}(x)\ \bar{b}(x) \,=\, \partial_2\phi(x)\ g'\big(\phi(x)\big)^2 \,=\, 0\,.$$

It means that g is a constant, say $g = -a \in \mathbb{R}$. In view of (3.14), one has

$$\exists\, \bar{k} \in C^\infty(\mathbb{R};\mathbb{R})\,; \qquad \bar{s}(x) \,=\, \bar{k}(b\,x_1 + x_2)\,.$$

Solve (3.1) to get

$$\Phi(t,x) \,=\, \psi(a\,t + b\,x_1 + x_2)\,.$$

It remains to look at (3.2) that is

$$\partial_t \mathbf{s} \,-\, a\ \partial_2 \mathbf{s} \,+\, \mathbf{s}\ \psi'\ (-\,\partial_1 + b\ \partial_2)\,\mathbf{s} \,=\, 0\,.$$

Observe that the choice

$$\mathbf{s}(t,x,\theta) \,=\, \bar{k}(z) \,+\, k^*\big(\psi(z),\theta\big)\ \psi'(z)\ {}^t(-1,b)\,, \qquad z = a\,t + b\,x_1 + x_2$$

is convenient. This is the expected result. □

Remark 3.2.1 - intuitive derivation of Lemma 3.3. The restriction on Φ contained in Lemma 3.3 can be guessed by looking at the construction underlying Lemma 2.7. To find simple waves, small amplitude terms (of size ε^j with $j \geq 1$) must be suppressed. Now, to get

$$\mathbf{u}^\varepsilon(0,x) \,=\, \check{h}^\varepsilon(x) \,=\, {}^t\big(\check{h}^{\varepsilon 1}(x), \check{h}^{\varepsilon 2}(x)\big) \,=\, H\big(x, \phi(x)/\varepsilon\big)$$

it is convenient to take

$$r\phi^\varepsilon \equiv 0\,, \qquad \phi_j \equiv 0\,, \qquad K_j \equiv 0\,, \qquad \forall\, j \in \{0, \cdots, J+1\}\,.$$

Then, consider $\check{h}^{\varepsilon 2}$. We must have

$$\partial_\theta \check{H}^2_0(x,z,\theta) \,=\, \partial_\theta \tilde{H}^2_0(z,\theta) \,=\, f_0(z)\ \partial_\theta K^*(z,\theta) \,+\, f'(z)\ K^*(z,\theta) \,=\, 0\,.$$

This is possible only if $f' \equiv 0$ on the set $\breve{\omega} = \operatorname{osup} K^* \neq \emptyset$. It follows that $g \equiv -a$ for some $a \in \mathbb{R}$ and, therefore, that $\Phi(t,x) = \psi(a\,t + b\,x_1 + x_2)$ for some function $\psi \in C^\infty(\mathbb{R};\mathbb{R})$. ◇

Remark 3.2.2 - weak convergence. The weak limit of the family $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ is

$$\check{\mathbf{u}}(t,x) \,=\, (\bar{k}\ \psi')(a\,t + b\,x_1 + x_2)\ {}^t(-1,b) \,-\, {}^t(0,a)\,.$$

Adjust the profile k so that

$$\bar{k}(z)^2 \,\not\equiv\, \textstyle\int_0^1\ k(z,\theta)^2\ \ d\theta\,.$$

Then, the weak limit of the product $\{\check{\mathbf{u}}^{\varepsilon 1}\,\check{\mathbf{u}}^{\varepsilon 2}\}_\varepsilon$ differs from the product of the weak limits, that is $\check{\mathbf{u}}^1\,\check{\mathbf{u}}^2$. A similar remark is made in [2] (see p. 495, example 12.9). Observe however that $\check{\mathbf{u}}(t,x)$ is still a solution to (1.1). Therefore, this argument does not bring any contradiction with the property of concentration-cancellation. ◇

4 The problem of stability.

Let (H^0, ϕ^0) be a couple which is compatible with (1.2) (see Lemma 2.5). Then, Lemma 3.1 furnishes a simple wave

$$\tilde{\mathbf{u}}^\varepsilon(t,x) \, = \, \mathbf{H}^0\big(t,x,\Phi^0(t,x)/\varepsilon\big)\,, \qquad \varepsilon \in \,]0,1]$$

which is a large amplitude wave $\tilde{\mathbf{u}}^\varepsilon$ issued from the initial data

$$\tilde{\mathbf{u}}^\varepsilon(0,x) \, = \, \tilde{h}^\varepsilon(x) \, := \, H^0\big(x,\phi^0(x)/\varepsilon\big)\,, \qquad \varepsilon \in \,]0,1]\,.$$

Now, consider the perturbed initial data $h^\varepsilon(x) = H^\varepsilon\big(x,\phi^0(x)/\varepsilon\big)$ where the profile H^ε is a small perturbation of H^0 having size $O(\varepsilon)$, that is

$$H^\varepsilon(x,\theta) \, := \, H^0(x,\theta) \, + \, \varepsilon\, W^\varepsilon(x,\theta) \, = \, H^0(x,\theta) \, + \, O(\varepsilon)\,. \tag{4.1}$$

Here, the profile $W^\varepsilon(x,\theta)$ is chosen of class C^∞ with respect to the variables $(\varepsilon,t,x,\theta) \in [0,1] \times \omega \times \mathbb{T}$. By virtue of subsection 2.2, the family $\{h^\varepsilon\}_\varepsilon$ is still well prepared on ω for (1.2) so that the oscillating Cauchy problem

$$\partial_t \mathbf{u}^\varepsilon \, + \, (\mathbf{u}^\varepsilon \cdot \nabla_x)\mathbf{u}^\varepsilon \, + \, \mathbf{f} \, = \, 0\,, \qquad \mathbf{u}^\varepsilon(0,x) \, = \, h^\varepsilon(x)\,, \qquad \varepsilon \in \,]0,1] \tag{4.2}$$

is well posed on Ω^τ for some $\tau > 0$. In other words, as explained in the introduction, the existence of a solution $\mathbf{u}^\varepsilon$ to (4.2) on a domain Ω^τ which does not shrink to the empty set (as ε goes to zero) is guaranteed *a priori.*

On the other hand, the analysis of chapters 2 and 3 gives no information about the asymptotic behavior on Ω^τ of $\{\mathbf{u}^\varepsilon\}_\varepsilon$. In order to see where the difficulty is, let us consider the linearized equation along $\tilde{\mathbf{u}}^\varepsilon$, that is

$$\partial_t \dot{\mathbf{u}} \, + \, (\tilde{\mathbf{u}}^\varepsilon \cdot \nabla_x)\dot{\mathbf{u}} \, + \, (\dot{\mathbf{u}} \cdot \nabla_x)\mathbf{H}^0 \, + \, \varepsilon^{-1} \, (\dot{\mathbf{u}} \cdot \nabla_x \Phi^0) \, \partial_\theta \mathbf{H}^0 \, = \, 0\,.$$

Multiplying on the left by ${}^t\dot{\mathbf{u}}$ and integrating over $\mathbb{R}^2$ yields the rough control

$$\tfrac{d}{dt} \, \| \, \dot{\mathbf{u}}(t,\cdot) \, \|^2_{L^2(\mathbb{R}^2)} \leq C \, \varepsilon^{-1} \; \| \, \dot{\mathbf{u}}(t,\cdot) \, \|^2_{L^2(\mathbb{R}^2)} \; .$$

This inequality does not provide with energy estimates which are uniform in $\varepsilon \in \,]0,1]$. It allows the presence of instabilities. Nothing guarantees that $\mathbf{u}^\varepsilon$ remains closed to $\tilde{\mathbf{u}}^\varepsilon$, and actually what follows shows that **this is not the case**. More precisely, the difference $\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}^\varepsilon$ is of size $O(\varepsilon)$ when $t = 0$ but it can become of size $O(1)$ for $t \in \,]0,1]$.

The possibility of such a mechanism of amplification is already hidden in the discussion of chapters 2 and 3. Indeed, there is a contrast between the weak conditions on (H,ϕ) in Lemma 2.7 and the strong restrictions given in Lemma 3.3. The first conditions (imposed on the *main* term of the oscillation) are weaker *because* small perturbations of size $O(\varepsilon)$ are allowed.

The preceding remark seems alleviating but it announces in fact the following more general principle :

When dealing with the propagation of a large amplitude wave which oscillates at the frequency $O(1/\varepsilon)$, the shape of the main contribution which is of size $O(1)$ is generically coupled with what happens at the level of smaller terms, for instance of size $O(\varepsilon)$.

The aim of this chapter 4 is to clarify this assertion. The links between large and small terms is cleared up in the next subsections.

4.1 Small perturbations with variations of the phase.

As explained in the introduction, a possible strategy is to look for a solution of the form (1.22). In the formula (1.22), both $\mathbf{H}^\varepsilon$ and Φ^ε are regarded as unknowns which can depend on ε. The introduction of Φ^ε corresponds to a *blow up* procedure. More precisely, we seek

$$\begin{aligned} &\mathbf{H}^\varepsilon(t,x,\tilde\theta) \in C^1(\Omega\times\mathbb{T};\mathbb{R}^2)\,, \qquad && \mathbf{H}^\varepsilon(t,x,\tilde\theta) \,=\, \mathbf{H}^0(t,x,\tilde\theta)+\varepsilon\,\mathbf{W}^\varepsilon(t,x,\tilde\theta)\,, \\ &\Phi^\varepsilon(t,x,\theta) \in C^1(\Omega\times\mathbb{T};\mathbb{R})\,, && \Phi^\varepsilon(t,x,\theta) \,=\, \Phi^0(t,x)+\varepsilon\,\mathbf{\Psi}^\varepsilon(t,x,\theta)\,, \end{aligned}$$

where $\mathbf{H}^0(t,x,\tilde\theta)$ and $\Phi^0(t,x)$ are given functions obtained through (1.12). We use the notation $\tilde\theta$ (instead of θ) to insist on the fact that the rapid variable $\tilde\theta$ must be replaced by $\Phi^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big)/\varepsilon$ $\big($instead of $\Phi^0(t,x)/\varepsilon\big)$. Therefore, we seek a solution $\mathbf{u}^\varepsilon$ having the form

$$\mathbf{u}^\varepsilon(t,x) \,=\, \mathbf{H}^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon+\mathbf{\Psi}^\varepsilon\big(t,x,\Phi^0(t,x)/\varepsilon\big)\big)\,, \qquad \varepsilon\in\,]0,1]\,. \tag{4.3}$$

Observe that, at the level of Φ^ε, the term $\mathbf{\Psi}^\varepsilon$ plays the part of a small perturbation of size $O(\varepsilon)$. But, at the level of $\mathbf{u}^\varepsilon$, it induces a large modification of size $O(1)$. At time $t=0$, we get

$$\begin{aligned} \mathbf{u}^\varepsilon(0,x) \,=\, H^\varepsilon\big(x,\phi^0(x)/\varepsilon\big) \,=\, &\mathbf{H}^0\big(0,x,\phi^0(x)/\varepsilon+\mathbf{\Psi}^\varepsilon\big(0,x,\phi^0(x)/\varepsilon\big)\big) \\ &+\varepsilon\,\mathbf{W}^\varepsilon\big(0,x,\phi^0(x)/\varepsilon+\mathbf{\Psi}^\varepsilon\big(0,x,\phi^0(x)/\varepsilon\big)\big)\,. \end{aligned}$$

This is coherent with (4.1) if we impose

$$\mathbf{W}^\varepsilon(0,x,\tilde\theta) \,=\, W^\varepsilon(x,\tilde\theta)\,, \qquad \mathbf{\Psi}^\varepsilon(0,x,\theta) \,=\, 0\,, \qquad \varepsilon\in\,]0,1]\,. \tag{4.4}$$

Plug $\mathbf{u}^\varepsilon$ given by (4.3) in the equation (1.2). Use (1.12) to make simplifications. It remains

$$\begin{aligned} \big[\,\partial_t\mathbf{W}^\varepsilon(t,x,\tilde\theta) \,+\, &\big((\mathbf{H}^\varepsilon(t,x,\tilde\theta)\cdot\nabla_x\big)\mathbf{W}^\varepsilon(t,x,\tilde\theta) \\ &+\big(\mathbf{W}^\varepsilon(t,x,\tilde\theta)\cdot\nabla_x\big)\mathbf{H}^0(t,x,\tilde\theta)\,\big]_{|\,\tilde\theta=\Phi^\varepsilon(t,x,\Phi^0(t,x)/\varepsilon)/\varepsilon} \end{aligned}$$

$$
\begin{aligned}
+\,\varepsilon^{-1}\,\Big[\,\partial_t\boldsymbol{\Psi}^\varepsilon(t,x,\theta) \,+\, &\big(\mathbf{H}^\varepsilon\big(t,x,\theta+\boldsymbol{\Psi}^\varepsilon(t,x,\theta)\big)\cdot\nabla_x\big)\boldsymbol{\Psi}^\varepsilon(t,x,\theta) \\
&+\big(\mathbf{W}^\varepsilon\big(t,x,\theta+\boldsymbol{\Psi}^\varepsilon(t,x,\theta)\big)\cdot\nabla_x\big)\Phi^0(t,x)\;\partial_\theta\boldsymbol{\Psi}^\varepsilon(t,x,\theta) \\
&+\big(\mathbf{W}^\varepsilon\big(t,x,\theta+\boldsymbol{\Psi}^\varepsilon(t,x,\theta)\big)\cdot\nabla_x\big)\Phi^0(t,x)\,\Big]_{|\,\theta=\Phi^0(t,x)/\varepsilon} \\
&\qquad\qquad\times\partial_{\tilde\theta}\mathbf{H}^\varepsilon\big(t,x,\Phi^\varepsilon(t,x,\Phi^0(t,x)/\varepsilon)/\varepsilon\big) \;=\; 0\,.
\end{aligned}
$$

In the above expression, we take care to precise which variables θ or $\tilde\theta$ are used. Since there are now two unknowns $\mathbf{W}^\varepsilon$ and $\boldsymbol{\Psi}^\varepsilon$, it is natural to incorporate a new constraint. We choose to add the equation which says that the singular term (with ε^{-1} in factor) vanishes (in fact classical arguments in geometric optics could be used to show that this equation is necessary). Hence, we will look for $\big(\mathbf{W}^\varepsilon(t,x,\tilde\theta),\boldsymbol{\Psi}^\varepsilon(t,x,\theta)\big)$ as a solution of the following nonlinear system

$$
\tag{4.5}
\left\{
\begin{aligned}
\partial_t\mathbf{W}^\varepsilon\ &+\!\big((\mathbf{H}^0+\varepsilon\,\mathbf{W}^\varepsilon)\cdot\nabla_x\big)\mathbf{W}^\varepsilon+(\mathbf{W}^\varepsilon\cdot\nabla_x)\mathbf{H}^0 \;=\; 0\,, \\
\partial_t\boldsymbol{\Psi}^\varepsilon\ &+\!\big((\mathbf{H}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\boldsymbol{\Psi}^\varepsilon \\
&\quad+\big(\mathbf{W}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\Phi^0\;\partial_\theta\boldsymbol{\Psi}^\varepsilon \\
&\quad+\big(\mathbf{W}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\Phi^0 \;=\; 0\,,
\end{aligned}
\right.
$$

where the first equation involves the variables $(t,x,\tilde\theta)$ whereas the second equation is in the variables (t,x,θ). The system (4.5) is completed with the initial data (4.4). It has a triangular structure. The first part writes

$$
\tag{4.6}
\left\{
\begin{aligned}
&\partial_t\mathbf{W}^\varepsilon+\big((\mathbf{H}^0+\varepsilon\,\mathbf{W}^\varepsilon)\cdot\nabla_x\big)\mathbf{W}^\varepsilon+(\mathbf{W}^\varepsilon\cdot\nabla_x)\mathbf{H}^0 \;=\; 0\,, \\
&\mathbf{W}^\varepsilon(0,x,\tilde\theta) \;=\; W^\varepsilon(x,\tilde\theta)\,.
\end{aligned}
\right.
$$

It is a standard initial value problem for a symmetric quasi-linear hyperbolic system, depending smoothly on $\varepsilon\in[0,1]$. It is locally well posed, uniformly in $\varepsilon\in[0,1]$, hence $\mathbf{W}^\varepsilon(t,x,\tilde\theta)$ is well defined by (4.6). The second part of (4.5) is

$$
\tag{4.7}
\left\{
\begin{aligned}
\partial_t\boldsymbol{\Psi}^\varepsilon &+\big(\mathbf{H}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\boldsymbol{\Psi}^\varepsilon \\
&\quad+\big(\mathbf{W}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\Phi^0(t,x)\;\partial_\theta\boldsymbol{\Psi}^\varepsilon \\
&\quad+\big(\mathbf{W}^\varepsilon(t,x,\theta+\boldsymbol{\Psi}^\varepsilon)\cdot\nabla_x\big)\Phi^0(t,x) \;=\; 0\,, \\
\boldsymbol{\Psi}^\varepsilon\,&(0,x,\theta) \;=\; 0\,.
\end{aligned}
\right.
$$

Since $\mathbf{W}^\varepsilon$ has been previously determined, this is a scalar quasi-linear equation depending smoothly on $\varepsilon\in[0,1]$. It yields $\boldsymbol{\Psi}^\varepsilon(t,x,\theta)$. When $\varepsilon=0$, the system (4.5) reduces to

$$
\tag{4.8}
\left\{
\begin{aligned}
\partial_t\mathbf{W}^0\ &+(\mathbf{H}^0\cdot\nabla_x)\mathbf{W}^0 \,+\, (\mathbf{W}^0\cdot\nabla_x)\mathbf{H}^0 \;=\; 0\,, \\
\partial_t\boldsymbol{\Psi}^0\ &+\big(\mathbf{H}^0(t,x,\theta+\boldsymbol{\Psi}^0)\cdot\nabla_x\big)\boldsymbol{\Psi}^0 \\
&+\big(\mathbf{W}^0(t,x,\theta+\boldsymbol{\Psi}^0)\cdot\nabla_x\big)\Phi^0\,\partial_\theta\boldsymbol{\Psi}^0 \\
&+\big(\mathbf{W}^0(t,x,\theta+\boldsymbol{\Psi}^0)\cdot\nabla_x\big)\Phi^0 \;=\; 0\,.
\end{aligned}
\right.
$$

It is completed with the initial conditions

$$(4.9) \qquad \mathbf{W}^0(0,x,\tilde{\theta}) \,=\, W^0(x,\tilde{\theta})\;, \qquad \mathbf{\Psi}^0(0,x,\theta) \,=\, 0\,.$$

The system (4.8) has a unique local smooth solution. In order to state the next result, fix $\tau_0 \in]0,\tau[$ and a solution $(\mathbf{W}^0, \mathbf{\Psi}^0)$ of (4.8)-(4.9) which is C^∞ on the domain $\Omega^\tau \times \mathbb{T}$.

Theorem 4.1. *Assume that* $(\mathbf{H}^0, \Phi^0)$ *is a smooth simple wave satisfying (1.12) on the domain* $\Omega^\tau \times \mathbb{T}$ *as constructed in Lemma 3.1, and let* $(H^0,\phi^0) = (\mathbf{H}^0|_{t=0}, \Phi^0|_{t=0})$ *the corresponding compatible initial value. For any family* $\{W^\varepsilon\}_\varepsilon$ *where* $W^\varepsilon(x,\theta)$ *is a smooth function with respect to* $(\varepsilon,x,\theta) \in [0,1] \times \omega \times \mathbb{T}$*, there exists* $\varepsilon_0 \in]0,1]$ *such that for all* $\varepsilon \in]0,\varepsilon_0]$ *the Cauchy problem (1.2) with initial data*

$$\mathbf{u}^\varepsilon(0,x) \,=\, H^0\big(x,\phi^0(x)/\varepsilon\big) \,+\, \varepsilon\; W^\varepsilon\big(x,\phi^0(x)/\varepsilon\big)$$

has a unique smooth solution $\mathbf{u}^\varepsilon(t,x) \in C^1(\Omega^{\tau_0})$*. Furthermore, we have the following asymptotic expansion*

$$\mathbf{u}^\varepsilon(t,x) \,=\, \mathbf{H}^0\big(t,x,\Phi^0(t,x)/\varepsilon + \mathbf{\Psi}^0\big(t,x,\Phi^0(t,x)/\varepsilon\big)\big) \,+\, O(\varepsilon)$$

which is verified as ε *goes to* 0 *in the sense of the space* $L^\infty(\Omega^{\tau_0})$*.*

Proof of Theorem 4.1.

1. Solve the first equation (4.6) to get $\mathbf{W}^\varepsilon$. Note that the only nonlinear term in this equation, which is $(\mathbf{W}^\varepsilon\cdot\nabla_x)\mathbf{W}^\varepsilon$, is multiplied by ε. Therefore, by taking ε small enough, the life span of the solution can be taken as close as we want to the life span of $\mathbf{H}^0$. Hence, for ε small enough $\mathbf{W}^\varepsilon$ is well defined and smooth on $\Omega^{\tau_0}\times\mathbb{T}$. Moreover, $\mathbf{W}^\varepsilon$ depends smoothly on ε.

2. Now, look at (4.7). This is a scalar quasi-linear equation. By smooth dependence on the parameter ε, the life span of $\mathbf{\Psi}^\varepsilon$ is (again) at least equal to τ_0 for $\varepsilon > 0$ small enough. Hence $\mathbf{\Psi}^\varepsilon$ is also a well defined smooth profile on $\Omega^{\tau_0}\times\mathbb{T}$, smooth in ε.

3. Again, since the function $(\mathbf{W}^\varepsilon, \mathbf{\Psi}^\varepsilon)$ depends smoothly on ε, the first term in the Taylor expansion with respect to ε is $(\mathbf{W}^0, \mathbf{\Psi}^0)$ and the asymptotic estimate follows. □

Remark 4.1.1 - comparison with the usual methods. At the level of the system (4.5), several similarities with the usual methods of geometrical optics are apparent: the first equation is a *transport equation* on the "profile" $\mathbf{W}^\varepsilon$ which propagates the amplitude of the wave, while the second equation is instead an *eiconal equation* on the "phase" $\mathbf{\Psi}^\varepsilon$, which brings informations related to the geometry of the propagation.

However, there is an inversion in the procedure since the profile $\mathbf{W}^\varepsilon$ is determined before the phase $\mathbf{\Psi}^\varepsilon$. Moreover, the term $\mathbf{\Psi}^\varepsilon$ could as well be considered as a contribution to the *amplitude* of the wave (instead of to the phase), just by introducing

$$\mathbf{U}^\varepsilon(t,x,\theta) \,:=\, \mathbf{H}^\varepsilon\big(t,x,\theta+\mathbf{\Psi}^\varepsilon(t,x,\theta)\big)\,. \tag{4.10}$$

This leads to the alternative representation (1.18) of the solution $\mathbf{u}^\varepsilon$. Thus, the status of both $\mathbf{W}^\varepsilon(t,x,\tilde{\theta})$ and $\mathbf{\Psi}^\varepsilon(t,x,\theta)$ is not perfectly clear. In the next subsection, to the sake of completeness, we work with the second point of view. In other words, the phase Φ^0 is kept fixed. It does not support variations as above. $\diamond$

4.2 Large amplitude modifications with fixed phase.

Again, we fix $\mathbf{H}$, Φ and $\mathbf{f}$ as in Lemma 3.1. We reset the rigidity of Φ which means to work only with (t,x,θ), by using the representation (1.18) where

$$\mathbf{U}^\varepsilon(t,x,\theta)\in C^1_b(\Omega^\tau\times\mathbb{T};\mathbb{R}^2)\,,\qquad \forall\,\varepsilon\in\,]0,1]\,.$$

The equation (1.2) becomes (1.19). It is completed with

$$\mathbf{U}^\varepsilon(0,x,\theta)\,=\,H^\varepsilon(x,\theta)\,=\,H(x,\theta)\,+\,\varepsilon\;W^\varepsilon(x,\theta)\,. \tag{4.11}$$

Since H^ε is smooth with respect to $\varepsilon\in[0,1]$, it can be expanded according to the Taylor formula

$$H^\varepsilon(x,\theta)\,\sim\,H(x,\theta)+\varepsilon\,\textstyle\sum_{j=0}^{+\infty}\,\varepsilon^j\;H_j(x,\theta)\,.$$

Of course, in view of paragraph 4.1, we know *a priori* that the Cauchy problem (1.19)-(4.11) can be solved on the domain $\Omega^\tau\times\mathbb{T}$. By the way, note that this is not a consequence of the analysis of chapter 2 which is not adapted to handle (1.19)-(4.11). Also, this is not a consequence of classical methods. Indeed, these methods rely on energy estimates related to the linearized equation along $\mathbf{H}(t,x,\theta)$, that is (1.20). Now, as before, this equation (1.20) leads to

$$\tfrac{d}{dt}\;\parallel\dot{\mathbf{U}}(t)\parallel^2_{L^2(\mathbb{R}^2\times\mathbb{T})}\leq\,C\;\varepsilon^{-1}\;\;\parallel\dot{\mathbf{U}}(t)\parallel^2_{L^2(\mathbb{R}^2\times\mathbb{T})}\;\;.$$

This differential equation is not sufficient to obtain some control which is uniform with respect to the parameter $\varepsilon\in\,]0,1]$. In fact, the structure of $\mathbf{H}^\varepsilon(t,x,\theta)$ is too rigid to absorb a perturbation of size $O(\varepsilon)$. This is the reason why we introduced more flexibility in the paragraph 4.1 by allowing variations in the phase. Our aim here is to present another version of this manipulation.

The influence of $\mathbf{\Psi}^\varepsilon$ is marked by a translation on the fast variable. This corresponds to the operation

$$\begin{array}{rcl} T_{\mathbf{V}^1} \, : \, C^1(\Omega^\tau \times \mathbb{T}) & \longrightarrow & C^1(\Omega^\tau \times \mathbb{T}) \\ \mathbf{H}(t,x,\theta) & \longmapsto & \mathbf{H}\big(t,x,\theta + \mathbf{V}^1\big) \end{array} \, , \qquad \mathbf{V}^1 \in \mathbb{R} \, .$$

The action $T_{\mathbf{V}^1}$ alters the profile $\mathbf{H}$ without changing the energy $\langle \mathbf{H} \cdot \mathbf{H} \rangle$ of the oscillation. From this point of view, the application $T_{\mathbf{V}^1}$ plays the part of a *gauge transformation*. Now, the idea is to incorporate $\mathbf{V}^1$ as a new state variable.

Proposition 4.1. *Select data $\mathbf{f}$ and H^ε as in (2.1) and (4.1). There is $\tau > 0$ such that, for all $\varepsilon \in]0,1]$, the Cauchy problem (1.19)-(4.11) has a solution $\mathbf{U}^\varepsilon$ on a fixed domain $\Omega^\tau \times \mathbb{T}$ (which does not shrink to the empty set as ε goes to zero).*

Moreover, there are smooth functions

$$\begin{aligned} X^\varepsilon &\equiv X(\varepsilon,t,x,\theta;\mathbf{V}) \ \in C^\infty\big([0,1] \times \Omega^\tau \times \mathbb{T} \times \mathbb{R}^3; \mathbb{R}^3\big) \, , \\ M^\varepsilon &\equiv M(\varepsilon,t,x,\theta;\mathbf{V}) \in C^\infty\big([0,1] \times \Omega^\tau \times \mathbb{T} \times \mathbb{R}^3; \mathrm{M}^3(\mathbb{R})\big) \, , \\ \Gamma^\varepsilon &\equiv \Gamma(\varepsilon,t,x,\theta;\mathbf{V}) \ \ \in C^\infty\big([0,1] \times \Omega^\tau \times \mathbb{T} \times \mathbb{R}^3; \mathbb{R}^2\big) \, , \end{aligned}$$

such that the solution $\mathbf{U}^\varepsilon$ of (1.19) can be represented as

$$\mathbf{U}^\varepsilon(t,x,\theta) \, = \, \Gamma\big(\varepsilon,t,x,\theta;\mathbf{V}^\varepsilon(t,x,\theta)\big) \, , \qquad \mathbf{V}^\varepsilon \, = \, {}^t(\mathbf{V}^{\varepsilon 1}, \mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3}) \tag{4.12}$$

where $\mathbf{V}^\varepsilon$ is subjected to the non singular Burgers type equation

$$\partial_t \mathbf{V}^\varepsilon \, + \, \big(X(\varepsilon,t,x,\theta;\mathbf{V}^\varepsilon) \cdot \nabla_{x,\theta}\big) \, \mathbf{V}^\varepsilon \, = \, M(\varepsilon,t,x,\theta;\mathbf{V}^\varepsilon) \, \mathbf{V}^\varepsilon \, . \tag{4.13}$$

We have $\mathbf{V}^\varepsilon = \mathbf{V}_0 + O(\varepsilon)$ with $\mathbf{V}_0 = {}^t(\mathbf{V}_0^1, 0, \mathbf{V}_0^3)$. The scalar components $\mathbf{V}_0^1$ and $\mathbf{V}_0^3$ are determined by solving the non linear system

$$\left\{ \begin{array}{l} \partial_t \mathbf{V}_0^1 \, + \, (\tilde{X}^0 \cdot \nabla_{x,\theta}) \mathbf{V}_0^1 \, = \, - \, |\nabla_x \Phi|^2 \ \mathbf{V}_0^3 \, , \\ \partial_t \mathbf{V}_0^3 \, + \, (\tilde{X}^0 \cdot \nabla_{x,\theta}) \mathbf{V}_0^3 \, = \, 0 \, , \end{array} \right. \tag{4.14}$$

associated with the initial data

$$\mathbf{V}_0^1(0,x,\theta) \, = \, 0 \, , \qquad \mathbf{V}_0^3(0,x,\theta) \, = \, |\nabla_x \phi(x)|^{-2} \ \nabla_x \phi(x) \cdot H_0(x,\theta)$$

and involving

$$\begin{aligned} \tilde{X}^0 \cdot \nabla_{x,\theta} \, &= \, \tilde{X}^0(t,x,\theta;\mathbf{V}_0^1,\mathbf{V}_0^3) \cdot \nabla_{x,\theta} \\ &:= \, H(t,x,\theta + \mathbf{V}_0^1) \cdot \nabla_x \, + \, |\nabla_x \Phi|^2 \ \mathbf{V}_0^3 \ \partial_\theta \, . \end{aligned}$$

Remark 4.2.1. - identification of the main term in the oscillation. The application Γ can be made explicit. Noting $\mathbf{V} = {}^t(\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3)$, it is

$$\Gamma(\varepsilon, t, x, \theta; \mathbf{V}) := \mathbf{H}(t, x, \theta + \mathbf{V}^1) + \mathbf{V}^2 \, \nabla_x \Phi(t,x)^\perp + \varepsilon \, \mathbf{V}^3 \, \nabla_x \Phi(t,x) .$$

It follows that

$$\mathbf{U}^\varepsilon(t,x,\theta) = \mathbf{H}\big(t, x, \theta + \mathbf{V}_0^1(t,x,\theta)\big) + O(\varepsilon) . \tag{4.15}$$

By construction, at time $t = 0$, the contribution $\mathbf{V}_0^1$ does not occur. Suppose that $\nabla_x \phi \cdot H_0 \not\equiv 0$. Then $\mathbf{V}_0^3(0,\cdot) \not\equiv 0$ and $\mathbf{V}_0^1$ becomes non trivial when $t > 0$, due to the coupling in system (4.14). We see here that an information of size $O(\varepsilon)$ at time $t = 0$ (namely $\varepsilon \, \mathbf{V}_0^3(0,\cdot) \, \nabla_x \phi$) can influence the shape $\mathbf{H}(t, x, \theta + \mathbf{V}_0^1)$ of the large amplitude wave when $t > 0$. $\diamond$

Remark 4.2.2 - other formulation of (4.14). Suppose that $\mathbf{f} \equiv 0$. Introduce the vector valued function $\mathbb{U} = {}^t(\mathbb{U}^1, \mathbb{U}^2, \mathbb{U}^3)$ defined according to

$${}^t(\mathbb{U}^1, \mathbb{U}^2) := \mathbf{H}(t, x, \theta + \mathbf{V}_0^1) , \qquad \mathbb{U}^3 := \big(1 + f(\Phi)^2\big) \, \mathbf{V}_0^3 .$$

The informations contained in (1.12) and (3.6) allow to extract from (4.14) the constraint

$$\partial_t \mathbb{U} + \mathbb{U}^1 \, \partial_1 \mathbb{U} + \mathbb{U}^2 \, \partial_2 \mathbb{U} + (\partial_2 \Phi)^2 \, \mathbb{U}^3 \, \partial_\theta \mathbb{U} = 0 \tag{4.16}$$

completed with

$$\mathbb{U}(0,x,\theta) = {}^t\big(H^1(x,\theta) \, , \, H^2(x,\theta) \, , \, \partial_2 \phi(x)^{-2} \, \nabla_x \phi \cdot H_0(x,\theta) \big) . \tag{4.17}$$

The equation (4.16) is a three dimensional *Burgers type* equation. Suppose that $\partial_2 \Phi$ is a function of Φ, say $\partial_2 \Phi = h(\Phi)$. This means that $f' \equiv g' \equiv 0$ or that, for all $t \in [0, \tau[$, the function $\Phi(t,\cdot)$ is constant on parallel lines (as in subsection 3.2). In this special case, the equation (4.16) can be further reduced. Just replace the component $\mathbb{U}^3$ by $h(\Phi)^2 \, \mathbb{U}^3$ to recover exactly a three dimensional Burger equation on $\mathbb{U}$. $\diamond$

Proof of Proposition 4.1. First, recall that

$$\nabla_x \Phi(t,x) = {}^t(\partial_1 \Phi, \partial_2 \Phi)(t,x) , \qquad \nabla_x \Phi(t,x)^\perp = {}^t(-\partial_2 \Phi, \partial_1 \Phi)(t,x) .$$

For each $(\varepsilon, t, x, \theta) \in]0,1] \times \Omega^\tau \times \mathbb{T}$, the application

$$\mathbb{R}^3 \ni \mathbf{V} \longmapsto \Gamma(\varepsilon, t, x, \theta; \mathbf{V}) \in \mathbb{R}^2$$

is surjective (but not injective). In particular, the initial data H^ε can be achieved as the image of some function V^ε. In other words

$$H^\varepsilon(x,\theta) = \Gamma\big(\varepsilon, 0, x, \theta; V^\varepsilon(x,\theta)\big) , \qquad V^\varepsilon = {}^t(V^{\varepsilon 1}, V^{\varepsilon 2}, V^{\varepsilon 3}) .$$

There are several possible choices for V^ε. Take $V^{\varepsilon 1} \equiv 0$ and seek

$$(V^{\varepsilon 2}, V^{\varepsilon 3}) \,=\, \textstyle\sum_{j=0}^{J} \varepsilon^j \, (V_j^2, V_j^3) + \varepsilon^{J+1} \, (r^{\varepsilon 2}, r^{\varepsilon 3}) \,, \qquad (r^{\varepsilon 2}, r^{\varepsilon 3}) \,=\, O(1) \,.$$

The contribution due to V_0^2 can be absorbed inside H. Thus, impose $V_0^2 \equiv 0$. Then, it suffices to set

$$\begin{aligned} V_j^2 \,&:=\, |\nabla_x \Phi|^{-2} \;\, \nabla_x \Phi^\perp \cdot H_{j-1} \,, \qquad && \forall \, j \in \{1, \cdots, J\} \,, \\ V_j^3 \,&:=\, |\nabla_x \Phi|^{-2} \;\, \nabla_x \Phi \cdot H_j \,, && \forall \, j \in \{0, \cdots, J-1\} \,, \end{aligned}$$

and then to adjust $(r^{\varepsilon 2}, r^{\varepsilon 3})$ conveniently. Observe that $V^\varepsilon \in C^1(\omega \times \mathbb{T}; \mathbb{R}^3)$ and that by construction the family $\{V^\varepsilon\}_\varepsilon$ is subjected to the uniform control

$$\sup \, \big\{ \parallel V^\varepsilon \parallel_{L^\infty(\omega \times \mathbb{T})} + \parallel D_x V^\varepsilon \parallel_{L^\infty(\omega \times \mathbb{T})} ; \; \varepsilon \in]0,1] \, \big\} \,<\, \infty \,. \tag{4.18}$$

At time $t = 0$, impose

$$\mathbf{V}^\varepsilon(0, x, \theta) \,=\, V^\varepsilon(x, \theta) \,, \qquad \forall \, \varepsilon \in]0,1] \,. \tag{4.19}$$

Seek a solution to (1.19) which can be expressed like in (4.12). This manipulation corresponds to a dependent change of state variable, the vector valued function $\mathbf{V}^\varepsilon(t, x, \theta)$ being the new unknown. Introduce the vector field X^ε which is such that

$$\begin{aligned} X^\varepsilon \cdot \nabla_{x,\theta} \,&=\, X(\varepsilon, t, x, \theta; \mathbf{V}) \cdot \nabla_{x,\theta} \,=\, X_x^\varepsilon \cdot \nabla_x \,+\, X_\theta^\varepsilon \, \partial_\theta \\ &=\, \mathbf{H}(t, x, \theta + \mathbf{V}^1) \cdot \nabla_x \,+\, \mathbf{V}^2 \; \nabla_x \Phi(t, x)^\perp \cdot \nabla_x \\ &\quad\; + \varepsilon \; \mathbf{V}^3 \; \nabla_x \Phi(t, x) \cdot \nabla_x \,+\, |\nabla_x \Phi(t, x)|^2 \; \mathbf{V}^3 \; \partial_\theta \,. \end{aligned}$$

For $\varepsilon = 0$, it remains

$$\begin{aligned} X^0 \cdot \nabla_{x,\theta} \,&\equiv\, X(0, t, x, \theta; \mathbf{V}) \cdot \nabla_{x,\theta} \\ &=\, \mathbf{H}(t, x, \theta + \mathbf{V}^1) \cdot \nabla_x \,+\, \mathbf{V}^2 \; \nabla_x \Phi(t, x)^\perp \cdot \nabla_x \,+\, |\nabla_x \Phi(t, x)|^2 \; \mathbf{V}^3 \; \partial_\theta \,. \end{aligned}$$

Use (1.12) to interpret (1.19) according to

$$\big(\partial_t \,+\, X(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon) \cdot \nabla_{x,\theta} \big) \; \Gamma\big(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon(t, x, \theta)\big) \,+\, \mathbf{f} \,=\, 0 \,.$$

Remember that $\mathbf{f}$ does not depend on θ. This fact and again (1.12) allow to reduce the preceding equation to

$$\begin{aligned} &\varepsilon^0 \; \big\{ \left(\partial_t + X^\varepsilon \cdot \nabla_{x,\theta} \right) \mathbf{V}^{\varepsilon 1} \,+\, |\nabla_x \Phi|^2 \; \mathbf{V}^{\varepsilon 3} \, \big\} \; \partial_\theta \mathbf{H} \\ &\quad + \varepsilon^0 \; \big\{ \left((\partial_t + X^\varepsilon \cdot \nabla_{x,\theta}) \, \mathbf{V}^{\varepsilon 2} \right) \, \nabla_x \Phi^\perp \,+\, \mathfrak{X}^{\varepsilon 2} \, \mathbf{V}^{\varepsilon 2} \, \big\} \\ &\quad + \varepsilon^1 \; \big\{ \left((\partial_t + X^\varepsilon \cdot \nabla_{x,\theta}) \, \mathbf{V}^{\varepsilon 3} \right) \, \nabla_x \Phi \,+\, \mathfrak{X}^{\varepsilon 3} \, \mathbf{V}^{\varepsilon 3} \, \big\} \,=\, 0 \,. \end{aligned}$$

Here the notation $\mathfrak{X}^{\varepsilon 2}$ is for the vector

$$\mathfrak{X}^{\varepsilon 2} \,:=\, \partial_t \nabla_x \Phi^\perp \,+\, (X_x^\varepsilon \cdot \nabla_x) \nabla_x \Phi^\perp \,+\, \big((\nabla_x \Phi^\perp \cdot \nabla_x) \mathbf{H} \big)(t, x, \theta + \mathbf{V}^{\varepsilon 1}) \,.$$

On the other hand, the notation $\mathfrak{X}^{\varepsilon 3}$ is for the vector

$$\mathfrak{X}^{\varepsilon 3} \,:=\, \partial_t \nabla_x \Phi \,+\, (X^\varepsilon_x \cdot \nabla_x) \nabla_x \Phi \,+\, \big((\nabla_x \Phi \cdot \nabla_x) \mathbf{H} \big)(t,x,\theta + \mathbf{V}^{\varepsilon 1}) \,.$$

There are three unknowns $\mathbf{V}^{\varepsilon 1}$, $\mathbf{V}^{\varepsilon 2}$ and $\mathbf{V}^{\varepsilon 3}$ for two equations so that the preceding constraint on $\mathbf{V}^\varepsilon$ is under-determined. To remedy to this, just impose the supplementary condition

$$X^\varepsilon \ \mathbf{V}^{\varepsilon 1} \,+\, |\nabla_x \Phi|^2 \ \mathbf{V}^{\varepsilon 3} \,=\, 0 \,.$$

Then, it remains (4.13) with

$$\begin{aligned} M^\varepsilon \,&=\, M(\varepsilon,t,x,\theta;\mathbf{V}^\varepsilon) \,=\, \big(M_{ij}(\varepsilon,t,x,\theta;\mathbf{V}^\varepsilon) \big)_{1 \leq i,j \leq 3} \\ &:= \,-\, \frac{1}{|\nabla_x \Phi|^2} \begin{pmatrix} 0 & 0 & |\nabla_x \Phi|^4 \\ 0 & \mathfrak{X}^{\varepsilon 2} \cdot \nabla_x \Phi^\perp & \varepsilon \ \mathfrak{X}^{\varepsilon 3} \cdot \nabla_x \Phi^\perp \\ 0 & \varepsilon^{-1} \ \mathfrak{X}^{\varepsilon 2} \cdot \nabla_x \Phi & \mathfrak{X}^{\varepsilon 3} \cdot \nabla_x \Phi \end{pmatrix} . \end{aligned}$$

We have to show that this matrix M^ε does not involve coefficients which are singular with respect to ε. In particular, we must pay attention in $M_{32}(\varepsilon,\cdot)$. Recall that

$$\mathbf{H} \,=\, \bar{\mathbf{H}} \,+\, \mathbf{s}^* \ \nabla_x \Phi(t,x)^\perp \,=\, \bar{\mathbf{H}} \,+\, \mathbf{s}^* \ \partial_2 \Phi \ \ {}^t\big(-1, f(\Phi) \big) \,.$$

Remark that

$$\begin{aligned} &\partial_t \nabla_x \Phi \,+\, (\bar{\mathbf{H}} \cdot \nabla_x) \, \nabla_x \Phi \,+\, (\nabla_x \Phi \cdot \nabla_x) \, \bar{\mathbf{H}} \,=\, (\partial_1 \bar{\mathbf{H}}^2 - \partial_2 \bar{\mathbf{H}}^1) \ \nabla_x \Phi^\perp \,, \\ &\partial_t \nabla_x \Phi^\perp \,+\, (\bar{\mathbf{H}} \cdot \nabla_x) \, \nabla_x \Phi^\perp \,+\, (\nabla_x \Phi^\perp \cdot \nabla_x) \, \bar{\mathbf{H}} \,=\, S \ \nabla_x \Phi \,, \end{aligned}$$

where S is the symmetric matrix

$$S \,=\, \begin{pmatrix} 2 \ \partial_2 \bar{\mathbf{H}}^1 & \partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1 \\ \partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1 & -\, 2 \ \partial_1 \bar{\mathbf{H}}^2 \end{pmatrix} .$$

The relation (3.7) gives rise to

$$\begin{aligned} {}^t\big(f(\Phi),1 \big) \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \bar{\mathbf{H}} \,&=\, (\nabla_x \Phi^\perp \cdot \nabla_x) \, \big[{}^t\big(f(\Phi),1 \big) \cdot \bar{\mathbf{H}} \big] \\ &=\, (\nabla_x \Phi^\perp \cdot \nabla_x) \, g(\Phi) = 0 \,. \end{aligned}$$

It follows that

$$\begin{aligned} {}^t\nabla_x \Phi \ S \ \nabla_x \Phi \,&=\, 2 \, \big[\, (\partial_1 \Phi)^2 \ \partial_2 \bar{\mathbf{H}}^1 + \partial_1 \Phi \ \partial_2 \Phi \ (\partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1) - (\partial_2 \Phi)^2 \ \partial_1 \bar{\mathbf{H}}^2 \, \big] \\ &=\, 2 \ \nabla_x \Phi \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \bar{\mathbf{H}} \,=\, 0 \,. \end{aligned}$$

Observe that

$$(\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi^\perp \,=\, -\, f'(\Phi) \ (\partial_2 \Phi)^2 \ \nabla_x \Phi^\perp \,.$$

The preceding informations can be collected in order to simplify the expressions given for $\mathfrak{X}^{\varepsilon 2}$ and $\mathfrak{X}^{\varepsilon 3}$. We find

$$\begin{aligned}
\mathfrak{X}^{\varepsilon 2} \, = \, & \varepsilon \, \mathbf{V}^{\varepsilon 3} \, (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp \\
& + \big\{ - f'(\Phi) \, (\partial_2 \Phi)^2 \, \big[2 \, \mathbf{s}^*(t,x,\theta + \mathbf{V}^{\varepsilon 1}) + \mathbf{V}^{\varepsilon 2} \big] \\
& \qquad + (\nabla_x \Phi^\perp \cdot \nabla_x) \mathbf{s}^* \, + \, |\nabla_x \Phi|^{-2} \ {}^t \nabla_x \Phi^\perp \ S \ \nabla_x \Phi \, \big\} \ \nabla_x \Phi^\perp \, , \\
\mathfrak{X}^{\varepsilon 3} \, = \, & \varepsilon \, \mathbf{V}^{\varepsilon 3} \, (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi \, + \, (\partial_1 \bar{\mathbf{H}}^2 - \partial_2 \bar{\mathbf{H}}^1) \ \nabla_x \Phi^\perp \\
& + \mathbf{s}^*(t,x,\theta + \mathbf{V}^{\varepsilon 1}) \, \big[\, (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp \, \big] \\
& + \mathbf{V}^{\varepsilon 2} \, (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi \, + \, (\nabla_x \Phi \cdot \nabla_x) \mathbf{s}^* \ \nabla_x \Phi^\perp \, .
\end{aligned}$$

It implies that

$$\varepsilon^{-1} \ \mathfrak{X}^{\varepsilon 2} \cdot \nabla_x \Phi \, = \, \mathbf{V}^{\varepsilon 3} \ (\partial_2 \Phi)^2 \ |\nabla_x \Phi|^2 \ f'(\Phi) \, .$$

Therefore

$$M^\varepsilon \, = \, (M^\varepsilon_{ij})_{1 \leq i,j \leq 3} \, = \, M^0 \, + \, \varepsilon \ M^1 \, + \, \varepsilon^2 \ M^2 \, , \qquad M^k \, = (M^k_{ij})_{1 \leq i,j \leq 3}$$

where the coefficients M^k_{ij} are smooth functions on $\Omega^\tau \times \mathbb{T} \times \mathbb{R}^3$. It means that M is a smooth function of $(t,x,\theta,\mathbf{V})$ and also of $\varepsilon \in [0,1]$. In particular, it satisfies uniform estimates with respect to $\varepsilon \in]0,1]$.

Consequently, the quasilinear symmetric system (4.13) is a nonlinear transport equation involving coefficients and a source term which all are non singular with respect to $\varepsilon \in [0,1]$. Therefore, it can be solved in the context of C^1 regularity by the usual method of characteristics, in a way similar to what has been done in paragraph 2.1.1.

Taking into account (4.18), there is a domain of determinacy $\Omega(\tau) \times \mathbb{T}$ such that, for all $\varepsilon \in]0,1]$, the Cauchy problem (4.13)-(4.19) has a solution $\mathbf{V}^\varepsilon$ on $\Omega^\tau \times \mathbb{T}$ (where τ does not depend on ε). By way of formula (4.12), we recover a solution $\mathbf{U}^\varepsilon$ to (1.19)-(4.11) on $\Omega^\tau \times \mathbb{T}$.

At time $t = 0$, the expression V^ε is given by the Taylor expansion

$$V^\varepsilon \, = \, \textstyle\sum_{j=0}^J \, \varepsilon^j \ V_j \, + \, \varepsilon^{J+1} \ r^\varepsilon \, , \qquad \forall \, \varepsilon \in]0,1]$$

where

$$V_j(x,\theta) \, = \, {}^t(V^1_j, V^2_j, V^3_j)(x,\theta) \, \in \, C^1(\omega \times \mathbb{T}; \mathbb{R}^3) \, , \qquad 0 \leq j \leq J \, .$$

Now, seek profiles

$$\mathbf{V}_j(t,x,\theta) \, = \, {}^t(\mathbf{V}^1_j, \mathbf{V}^2_j, \mathbf{V}^3_j)(t,x,\theta) \, \in \, C^1\big(\Omega^\tau \times \mathbb{T}; \mathbb{R}^3\big) \, , \qquad 0 \leq j \leq J$$

adjusted so that the expression

$$\mathbf{V}^\varepsilon_a(t,x,\theta) \, = \, \textstyle\sum_{j=0}^J \varepsilon^j \ \mathbf{V}_j(t,x,\theta) \, , \qquad \varepsilon \in]0,1] \tag{4.20}$$

is an approximate solution of (4.13) in the following sense

$$\sup \ \big\{ \varepsilon^{-J} \ \ \| \mathbf{V}^\varepsilon - \mathbf{V}_a^\varepsilon \|_{C^0(\Omega^\tau \times \mathbb{T})} \, ; \ \varepsilon \in]0,1] \, \big\} \ < \ \infty \, .$$

The vector valued functions $\mathbf{V}_j$ are determined as follows. Substitute the sum (4.20) into (4.13) to get a formal expansion. The term with ε^j in factor can be put in the form

$$(\mathcal{M}_j) \qquad \partial_t \mathbf{V}_j \, + \, (X^0 \cdot \nabla_{x,\theta}) \mathbf{V}_j \, + \, \mathcal{F}_j(t,x,\theta; \mathbf{V}_0, \cdots, \mathbf{V}_j) \, = \, 0$$

where $X^0 = X(0,t,x,\theta;\mathbf{V}_0)$. Complete $(\mathcal{M}_j)$ with the initial data

$$(4.21) \qquad \mathbf{V}_j(0,x,\theta) \, = \, V_j(x,\theta) \, .$$

In particular, to identify the main contribution $\mathbf{V}_0$ in $\mathbf{V}^\varepsilon$, it suffices to solve

$$\partial_t \mathbf{V}_0 \, + \, (X^0 \cdot \nabla_{x,\theta}) \mathbf{V}_0 \, = \, M^0 \, \mathbf{V}_0 \, , \qquad \mathbf{V}_0(0,x,\theta) \, = \, {}^t(0,0,V_0^3) \, .$$

Since $M_{23}^0 \equiv 0$, the condition $\mathbf{V}_0^2(0,\cdot) \equiv 0$ is propagated which means that

$$\mathbf{V}_0(t,x,\theta) \, = \, {}^t\big(\mathbf{V}_0^1(t,x,\theta), 0, \mathbf{V}_0^3(t,x,\theta)\big) \, , \qquad \forall \, (t,x,\theta) \in \Omega^\tau \times \mathbb{T} \, .$$

and which implies that $X^0 \equiv \tilde{X}^0$. It remains to compute

$$\begin{aligned} M_{33}^0\big(t,x,\theta; {}^t(\mathbf{V}^1, 0, \mathbf{V}^3)\big) \, &= \, \mathbf{s}^*(t,x,\theta + \mathbf{V}^{\varepsilon 1}) \\ & \quad \times \, \nabla_x \Phi \cdot \big[\, (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp \, \big] \, . \end{aligned}$$

On the one hand

$$\nabla_x \Phi \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi \, = \, - \, (\partial_2 \Phi)^2 \ |\nabla_x \Phi|^2 \ f'(\Phi) \, .$$

On the other hand

$$\nabla_x \Phi \cdot (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp \, = \, \partial_2 \Phi \ \nabla_x \Phi \cdot {}^t\big(0, f'(\Phi) \ |\nabla_x \Phi|^2\big) \, .$$

Therefore

$$M_{33}^0\big(t,x,\theta; {}^t(\mathbf{V}^1, 0, \mathbf{V}^3)\big) \, = \, 0$$

and the validity of (4.14) is established. Once $\mathbf{V}_0$ has been identified, the systems $(\mathcal{M}_j)$ with $0 < j$ are made of *linear* transport equations. Therefore, the Cauchy problems $(\mathcal{M}_j)$-(4.21) with $0 < j$ can be solved inductively on the whole domain of determinacy $\Omega(\tau) \times \mathbb{T}$ where $\mathbf{V}_0$ is defined. □

Remark 4.2.3 - the incidence of the procedure on θ. At the level of (4.12), the status of θ is changed. There, it plays the part of a *slow* variable. ◇

Remark 4.2.4 - come back to the original state variables. The expression

$$\mathbf{u}^\varepsilon(t,x) \, = \, \mathbf{U}^\varepsilon\big(t,x,\Phi(t,x)/\varepsilon\big) \, , \qquad \varepsilon \in]0,1]$$

where $\mathbf{U}^\varepsilon$ is given by Proposition 4.1 is a solution to (4.2). ◇

Remark 4.2.5 - how the initial data $\mathbf{V}^\varepsilon(0,\cdot)$ *have been yet adjusted.* In order to deal with small $O(\varepsilon)$ perturbations of $\tilde{\mathbf{u}}^\varepsilon(0,\cdot)$, subsection 4.1 selects the choices $\mathbf{V}^{\varepsilon 1}(0,\cdot)\equiv 0$ and $\mathbf{V}^{\varepsilon 2}(0,\cdot)=O(\varepsilon)$. In the present paragraph 4.2, we impose $\mathbf{V}^{\varepsilon 1}(0,\cdot)\equiv 0$ and $\mathbf{V}^{\varepsilon 2}(0,\cdot)=O(1)$. This is more general. $\diamond$

Remark 4.2.6 - how the initial data $\mathbf{V}^\varepsilon(0,\cdot)$ *can be further adjusted.* In Proposition 4.1, there is some overlap of unknowns between $\mathbf{V}^{\varepsilon 1}$ and $\mathbf{V}^{\varepsilon 2}$. For instance, at time $t=0$, we can fix $\mathbf{V}^\varepsilon(0,\cdot)$ so that

$$\partial_\theta \left[H\big(x,\theta+\mathbf{V}^{\varepsilon 1}(0,x,\theta)\big)+\mathbf{V}^{\varepsilon 2}(0,x,\theta)\ \nabla_x\phi(x)^\perp \right] \equiv 0\,.$$

This means that we start with large amplitude well polarized oscillating modifications of the basic solution $H_\perp\big(\phi(x)\big)$. $\diamond$

Remark 4.2.7 - which profiles must be translated by $T_{\mathbf{V}^1}$*.* In Proposition 4.1, the action $T_{\mathbf{V}^1}$ is applied only on the main profile $\mathbf{H}$. It is not reported on the other terms $\mathbf{H}_j$ of the asymptotic expansion associated with $\mathbf{H}^\varepsilon$. This is certainly the source of some technical complications in the computations reported above. $\diamond$

The manipulations of subsections 4.1 and 4.2 are interesting because they indicate how the various ingredients are coupled. Put together, they contain the optimal and more flexible version of the technicalities proposed here. Now, to get a rapid and simple reply to some specific question, it is necessary to implement the preceding tools with precautions.

4.3 The two points of view conciliated.

In view of subsections 4.1 and 4.2, there are different manners to handle the problem of stability. Either we can introduce $\mathbf{\Psi}^\varepsilon$ or not. If not, we still have a reply by appealing to the Proposition 4.1 which includes a more general situation. But even at the level of this Proposition 4.1, it remains some flexibility (see remarks 4.2.5, 4.2.6 and 4.2.7).

In some sense, we can say that the more suitable version of the stability discussion is the one which compounds the preceding arguments in order to get the more elegant and direct proof. This is what we propose here. Seek a solution of (1.2) having the form

$$\begin{aligned} &\mathbf{u}^\varepsilon(t,x)=\mathbf{H}^\varepsilon\big(t,x,\Phi_c^\varepsilon(t,x)/\varepsilon\big)\,, \qquad &&\mathbf{H}^\varepsilon(t,x,\tilde\theta)\in C^\infty(\omega\times\mathbb{T};\mathbb{R}^2)\,,\\ &\Phi_c^\varepsilon(t,x)=\Phi^\varepsilon\big(t,x,\Phi(t,x)/\varepsilon\big)\,, \qquad &&\Phi^\varepsilon(t,x,\theta)\in C^\infty(\omega\times\mathbb{T};\mathbb{R})\,, \end{aligned} \tag{4.22}$$

where both $\mathbf{H}^\varepsilon$ and Φ^ε are regarded as unknowns.

Despite this apparent complexity, simple equations can be imposed on $\mathbf{H}^\varepsilon$ and Φ^ε in order to recover a solution $\mathbf{u}^\varepsilon$ of (1.2). It suffices to take

$$\partial_t \mathbf{H}^\varepsilon + (\mathbf{H}^\varepsilon \cdot \nabla_x)\mathbf{H}^\varepsilon + \mathbf{f} = 0\,, \qquad \mathbf{H}^\varepsilon(0,\cdot) = H^\varepsilon\,, \tag{4.23}$$

$$\partial_t \Phi_c^\varepsilon + \big(\mathbf{H}^\varepsilon(t,x,\Phi_c^\varepsilon/\varepsilon)\cdot\nabla_x\big)\Phi_c^\varepsilon = 0\,, \qquad \Phi_c^\varepsilon(0,\cdot) = \phi\,. \tag{4.24}$$

Like in subsection 4.1, the equation (4.23) is decoupled from (4.24). It can be handled apart. It involves the variables t, x and $\tilde\theta$. But $\tilde\theta$ can be regarded as a parameter. Select its solution $\mathbf{H}^\varepsilon(t,x,\tilde\theta)$ and look at (4.24). At first sight, the solution Φ_c^ε of (4.24) can develop shocks within time $O(\varepsilon)$ or even before. However, such a rapid formation of singularities does not occur. The reason is that $\mathbf{H}^\varepsilon$ can be decomposed in the following way

$$\begin{aligned}\mathbf{H}^\varepsilon(t,x,\tilde\theta) = {} & H_\perp(\Phi) + \mathbb{W}^{\varepsilon 1}(t,x,\tilde\theta)\ {}^t\big(-1,f(\Phi)\big) \\ & + \varepsilon\ \mathbb{W}^{\varepsilon 2}(t,x,\tilde\theta)\ {}^t\big(f(\Phi),1\big)\,.\end{aligned} \tag{4.25}$$

Now, seek a solution Φ_c^ε of (4.24) with the form

$$\Phi_c^\varepsilon(t,x) = \Phi(t,x) + \varepsilon\ \mathbf{\Psi}^\varepsilon\big(t,x,\Phi(t,x)/\varepsilon\big)\,. \tag{4.26}$$

Recall (3.6) and the relations

$$\partial_t\Phi + (\bar{\mathbf{H}}\cdot\nabla_x)\Phi = \partial_t\Phi + \big(H_\perp(\Phi)\cdot\nabla_x\big)\Phi = \partial_t\Phi + g(\Phi)\ \partial_2\Phi = 0\,.$$

Plug (4.26) in (4.24). It remains the following non singular equation

$$\begin{aligned}\partial_t\mathbf{\Psi}^\varepsilon & + \big(\mathbf{H}^\varepsilon(t,x,\theta+\mathbf{\Psi}^\varepsilon)\cdot\nabla_x\big)\mathbf{\Psi}^\varepsilon \\ & + \big(f(\Phi)^2+1\big)\ \partial_2\Phi\ \mathbb{W}^{\varepsilon 2}(t,x,\theta+\mathbf{\Psi}^\varepsilon)\ (\partial_\theta\mathbf{\Psi}^\varepsilon+1) = 0\end{aligned} \tag{4.27}$$

which is associated with the initial data $\mathbf{\Psi}^\varepsilon(0,\cdot)\equiv 0$. Now, we have to check that (4.25) is compatible with (4.23). Consider $\mathbb{W}^\varepsilon := {}^t(\mathbb{W}^{\varepsilon 1},\mathbb{W}^{\varepsilon 2})$ as a new unknown. Then use the property (Lemma 3.1)

$$\exists\, \mathbf{k}\in C^1\big(\Omega^\tau;\mathbb{R}\big)\,; \qquad \mathbf{f}(t,x) = -\,\mathbf{k}(t,x)\ {}^t\big(-1,f(\Phi)\big)$$

and the relation (1.15) to extract the system

$$\left\{\begin{aligned}\partial_t\mathbb{W}^{\varepsilon 1} + (\mathbf{H}^\varepsilon\cdot\nabla_x)\mathbb{W}^{\varepsilon 1} &= \mathbf{k} + \varepsilon\ \mathbb{W}^{\varepsilon 2}\ L^{\varepsilon 1}(t,x,\mathbb{W}^\varepsilon)\,, \\ \partial_t\mathbb{W}^{\varepsilon 2} + (\mathbf{H}^\varepsilon\cdot\nabla_x)\mathbb{W}^{\varepsilon 2} &= \mathbb{W}^{\varepsilon 2}\ L^{\varepsilon 2}(t,x,\mathbb{W}^\varepsilon)\,,\end{aligned}\right. \tag{4.28}$$

built with

$$L^{\varepsilon 1}(t,x,\mathbb{W}^\varepsilon) = f'(\Phi)\ \partial_2\Phi\ \Big[\,\tfrac{g}{1+f^2}(\Phi) - f(\Phi)\ \mathbb{W}^{\varepsilon 1} + \varepsilon\ \mathbb{W}^{\varepsilon 2}\,\Big]\,,$$

$$L^{\varepsilon 2}(t,x,\mathbb{W}^\varepsilon) = \partial_2\Phi\ \Big[\,\tfrac{g\,f\,f'}{1+f^2}(\Phi) - g'(\Phi) - f'(\Phi)\ \mathbb{W}^{\varepsilon 1} - \varepsilon\ f(\Phi)\ f'(\Phi)\ \mathbb{W}^{\varepsilon 2}\,\Big]\,.$$

Observe that the functions $L^{\varepsilon 1}$ and $L^{\varepsilon 2}$ are both smooth with respect to $(\varepsilon, t, x, \mathbb{W}) \in [0,1] \times \Omega^\tau \times \mathbb{R}^2$. Complete (4.28) with the initial data

$$
\begin{aligned}
\mathbb{W}^{\varepsilon 1}(0, x, \tilde{\theta}) \, &= \, r(x, \tilde{\theta}) \, + \, \varepsilon \, \big(1 + f(\phi)^2\big)^{-1} \ {}^t\big(-1, f(\phi)\big) \cdot W^\varepsilon \, , \\
\mathbb{W}^{\varepsilon 2}(0, x, \tilde{\theta}) \, &= \, \big(1 + f(\phi)^2\big)^{-1} \ {}^t\big(f(\phi), 1\big) \cdot W^\varepsilon \, .
\end{aligned} \tag{4.29}
$$

Since (4.28) contains no singularity, the Cauchy problem (4.28)-(4.29) is well posed on a domain $\Omega^\tau \times \mathbb{T}$ which does not depend on $\varepsilon \in]0,1]$. Moreover, the second equation in (4.28) preserves the size of $\mathbb{W}^{\varepsilon 2}$. For instance, if $\mathbb{W}^{\varepsilon 2}(0, \cdot) \equiv 0$, we still have $\mathbb{W}^{\varepsilon 2} \equiv 0$.

By the Taylor formula, $\mathbf{\Psi}^\varepsilon$ and $\mathbb{W}^\varepsilon$ can be expanded according to

$$
\mathbf{\Psi}^\varepsilon \, \sim \, \textstyle\sum_{j=0}^\infty \, \varepsilon^j \, \mathbf{\Psi}_j \, , \qquad \mathbb{W}^\varepsilon \, \sim \, \sum_{j=0}^\infty \, \varepsilon^j \, \mathbb{W}_j \, , \qquad \mathbb{W}_j \, = \, {}^t(\mathbb{W}_j^1, \mathbb{W}_j^2) \, .
$$

The expression $\mathbb{W}_0$ is obtained by looking at

$$
\left\{
\begin{aligned}
& \partial_t \mathbb{W}_0^1 + \big(H_\perp(\Phi) \cdot \nabla_x\big) \mathbb{W}_0^1 \\
& \qquad + \mathbb{W}_0^1 \, \big(-\partial_1 + f(\Phi) \, \partial_2\big) \mathbb{W}_0^1 \, = \, \mathbf{k} \, , \\
& \partial_t \mathbb{W}_0^2 + \big(H_\perp(\Phi) \cdot \nabla_x\big) \mathbb{W}_0^2 \\
& \qquad + \mathbb{W}_0^1 \, \big(-\partial_1 + f(\Phi) \, \partial_2\big) \mathbb{W}_0^2 \, = \, \mathbb{W}_0^2 \, L^{02}(t, x, \mathbb{W}_0) \, ,
\end{aligned}
\right. \tag{4.30}
$$

completed with

$$
\mathbb{W}_0^1(x, \tilde{\theta}) \, = \, r(x, \tilde{\theta}) \, , \qquad \mathbb{W}_0^2(x, \tilde{\theta}) \, = \, \big(1 + f(\phi)^2\big)^{-1} \ {}^t\big(f(\phi), 1\big) \cdot W^0(x, \tilde{\theta}) \, .
$$

In particular, we recover $\mathbb{W}_0^1 \equiv \mathbf{r}$ which means implicitly that the oscillating part $\mathbf{H}^*$ of the simple wave $\tilde{\mathbf{u}}^\varepsilon$ has been incorporated here as an unknown (instead of being fixed as before). On the other hand, the function $\mathbf{\Psi}_0$ is obtained through the condition $\mathbf{\Psi}_0(0, \cdot) \equiv 0$ and the equation

$$
\begin{aligned}
& \partial_t \mathbf{\Psi}_0 \, + \big(\mathbf{H}(t, x, \theta + \mathbf{\Psi}_0) \cdot \nabla_x\big) \mathbf{\Psi}_0 \\
& \qquad + \big(f(\Phi)^2 + 1\big) \, \partial_2 \Phi \, \mathbb{W}_0^2(t, x, \theta + \mathbf{\Psi}_0) \, (\, \partial_\theta \mathbf{\Psi}_0 + 1) \, = \, 0 \, .
\end{aligned} \tag{4.31}
$$

We recover that $\mathbf{U}^\varepsilon(t, x, \theta) = \mathbf{U}_0(t, x, \theta) + O(\varepsilon)$ with

$$
\begin{aligned}
\mathbf{U}_0(t, x, \theta) \, &= \, H_\perp(\Phi) \, + \, \mathbf{r}\big(t, x, \theta + \mathbf{\Psi}^0(t, x, \theta)\big) \ {}^t\big(-1, f(\Phi)\big) \\
&= \, \mathbf{H}\big(t, x, \theta + \mathbf{\Psi}_0(t, x, \theta)\big) \, .
\end{aligned}
$$

Remark 4.3.1 - instability of the profile. Suppose that $W^0 \cdot \nabla_x \phi \not\equiv 0$. Then, in view of (4.31), we have $\mathbf{\Psi}_0 \not\equiv 0$ so that $\mathbf{U}_0 \not\equiv \mathbf{H}$. We see here that a perturbation of $\mathbf{U}^\varepsilon(0, \cdot)$ of size $O(\varepsilon)$ at time $t = 0$ can induce at a time $t > 0$ a modification of $\mathbf{U}^\varepsilon(t, \cdot)$ of size $O(1)$. The nonlinear mapping $(H, H_0, \cdots, H_J) \longmapsto \mathbf{U}_0(t, \cdot)$ does depend on H_0. This explains why the main profile $\mathbf{U}_0(t, \cdot)$ cannot be deduced only from $\mathbf{U}_0(0, \cdot) \equiv H$.

By extension, to set a Cauchy problem in order to determine the Young measure associated with the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$ makes no sense. Again, this can be understood as a nonlinear instability result of the solution $\mathbf{H}$ of the original profile equation (1.19). $\diamond$

Remark 4.3.2 - in short. The manipulations (4.22) and (4.25) are the combinations of two things. First, in (4.22), the introduction of a new state variable $\mathbf{\Psi}^\varepsilon$ (this is not strictly required but this simplifies the analysis). Secondly, in (4.25), a change of state variables which is singular in $\varepsilon \in]0,1]$.

A similar argument has been already used in [9] to deal with large amplitude high frequency waves polarized on the entropy. It can be implemented here at the level of the speed $\mathbf{V}^\varepsilon$ because the basic solution $H_\perp(\Phi)$ and the phase Φ have both a very special structure. $\diamond$

5 Applications.

This final Section is devoted to consequences of the preceding analysis.

5.1 Various caustics phenomena for Euler equations

Lemmas 2.7 and 2.8 explain how to adjust the data H^ε (or V^ε or W^ε) and ϕ in order to have (1.33). Lemma 2.4 says that the corresponding solution $\check{\mathbf{u}}^\varepsilon(t,x)$ to (1.2) where $\mathbf{f} \equiv 0$ is also a solution to (1.1) with $\mathbf{p}$ constant. Then, Proposition 4.1 exhibits for the subsequent family $\{\check{\mathbf{U}}^\varepsilon\}_\varepsilon$ the asymptotic behaviour (4.15).

Note however that Lemmas 2.7 and 2.8 require conditions on H and H_0. These conditions can influence $\mathbf{H}$ and $\mathbf{V}_0^1$. In other words, the formula (4.15), when applied in the context (1.1), involves special expressions $\mathbf{H}$ and $\mathbf{V}_0$. The discussion starts by examining and commenting the constraints thus retrieved on $\mathbf{H}$ and $\mathbf{V}_0^1$.

In contrast with the situation of Lemma 3.3, Lemma 2.7 demands no particular restriction on the phase ϕ, except (2.20). Moreover, the function Φ is obtained through (3.1) where g is given by (2.34). As noted in remark 3.1.4, the phase Φ can develop shocks.

This formation of shocks corresponds to caustics phenomena of the *first* type for (1.1). This can surprise since Euler equations are often regarded as being “linearly degenerate”. But, it is so. Note however that the analysis of stability in Section 4 stops before the formation of these caustics.

The profile H must be subjected to (2.33). The corresponding expression $\mathbf{H}$ can be deduced from (3.2) or (3.10). We find

$$\mathbf{H} \equiv \breve{\mathbf{H}} := K(\Phi, \theta + \Phi_0) \ {}^t(1, -f \circ \Phi) + {}^t(0, g \circ \Phi) \tag{5.1}$$

where the scalar function $\Phi_0(t,x,\theta)$ is determined by

$$\begin{cases} \partial_t \Phi_0 + K(\Phi, \theta + \Phi_0) \ (\partial_1 - f \circ \Phi \ \partial_2) \, \Phi_0 + g \circ \Phi \ \partial_2 \Phi_0 = 0 \,, \\ \Phi_0(0, \cdot) = \phi_0 \,. \end{cases} \tag{5.2}$$

The expression Φ_0 which appears here plays an ambiguous part. Of course, it is linked with the underlying scalar profile $\mathbf{r}$ and it influences H^* (since it can depend on θ in a non trivial way). But also, it acts on K as a phase shift. To insist on this second aspect, Φ_0 is called the *second phase*.

Now, consider the following specific situation. Take

$$\omega = \mathbb{R}^2 \,, \qquad f \equiv g \equiv 0 \,, \qquad \phi(x) = \Phi(t,x) = x_2 \,.$$

Work in the framework of Lemma 2.7. Impose $\phi_{0|x_1=0} \equiv 0$ but $f_0 \not\equiv 0$ so that $\phi_0(x) = -x_1 \, f_0(x_2)$. Select a profile K satisfying $K(0,0) = 0$ and $\bar{K} \equiv 0$ (which is compatible with the choice $g \equiv 0$) but $K^* \not\equiv 0$. It remains

$$H(x,\theta) = K^*\big(x_2, \theta - x_1 \, f_0(x_2)\big) \ {}^t(1,0) \,, \qquad 0 \not\equiv f_0 \in C_b^\infty(\mathbb{R};\mathbb{R}) \,.$$

Under these conditions, the Cauchy problem (5.2) reduces to

$$\partial_t \Phi_0 + K^*(x_2, \theta + \Phi_0) \ \partial_1 \Phi_0 = 0 \,, \qquad \Phi_0(0,\cdot) = -\, x_1 \ f_0(x_2) \,.$$

The variables x_2 and θ can be regarded as parameters. Since by construction

$$\exists \, (x_1, x_2, \theta) \in \mathbb{R}^2 \times \mathbb{T} \,; \qquad - f_0(x_2) \ \partial_\theta K^*\big(x_2, \theta - x_1 \, f_0(x_2)\big) < 0$$

shocks do appear at the level of Φ_0. In other words

$$\exists \, \tilde{T} \in \mathbb{R}_+ \,; \qquad \lim_{t \longrightarrow \tilde{T}-} \ \| \, \partial_1 \Phi_0(t,\cdot) \, \|_{L^\infty(\mathbb{R}^2 \times \mathbb{T})} = +\infty \,.$$

By virtue of Lemma 2.7, the couple (H, ϕ) which is selected above is associated with the *main term* of oscillations $\{\breve{h}^\varepsilon\}_\varepsilon$ which are well prepared on $\mathbb{R}^2$ for (1.1). More precisely

$$\breve{h}^\varepsilon(x) = H\big(x, \phi(x)/\varepsilon\big) + \varepsilon \, rh^\varepsilon(x) \in \mathcal{V}_0^\tau(\mathbb{R}^2) \,, \qquad rh^\varepsilon \not\equiv 0 \,, \qquad \forall \, \varepsilon \in]0,1] \,.$$

Note $\breve{\mathbf{u}}^\varepsilon(t,x)$ the solution issued from $\breve{h}^\varepsilon$ by solving (1.2) with $\mathbf{f} \equiv 0$. Applying Lemma 2.4 (see also remark 2.1.6), the function $\breve{\mathbf{u}}^\varepsilon$ is a *global* (in both time and space variables) C^1 solution to (1.1).

Applying Lemma 3.1, the couple (H, ϕ) gives rise also to a simple wave $\tilde{\mathbf{u}}^\varepsilon$ which is a *local* solution to (1.2) with $\mathbf{f} \equiv 0$, defined on the domain $[0, \tilde{T}[\times \mathbb{R}^2$, with *breakdown* when $t < \tilde{T}$ goes to $\tilde{T}$. We have seen in Section 4 that the asymptotic behaviour of the family $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ is modeled by the one of $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$ (modulo some translation in the fast variable θ). There is no contradiction between the two preceding assertions :

- On the one hand, looking at Φ_0 as a second phase, the presence of shocks can be interpreted as *caustics* phenomena of the *second* type for Euler equations. It does not preclude the solution $\check{\mathbf{u}}^\varepsilon$ to remain smooth. In fact, only from this point of view, the situation can be compared to what is observed in the framework of semilinear equations [18].

- On the other hand, the simple wave $\tilde{\mathbf{u}}^\varepsilon$ is a solution of (1.2) but it is not a solution of (1.1). Indeed, the profile $H(x,\theta)$ does not depend only on x_2, as is required in Lemma 3.3. Thus, nothing guarantees that $\tilde{\mathbf{u}}^\varepsilon$ remains a smooth function.

5.2 On the concentration-cancellation property.

We start this paragraph by briefly recalling what is foreseen by homogenization techniques. Seek a solution of (1.1) with

$$\begin{aligned} \mathbf{u}^\varepsilon(t,x) \,&=\, \mathcal{H}\big(t,x,\Phi(t,x)/\varepsilon\big) \,+\, \varepsilon\, \mathcal{H}_0\big(t,x,\Phi(t,x)/\varepsilon\big) \,+\, O(\varepsilon^2)\,, \\ \mathbf{p}^\varepsilon(t,x) \,&=\, \mathbf{p}(t,x) \,+\, \varepsilon\, P_0\big(t,x,\Phi(t,x)/\varepsilon\big) \,+\, O(\varepsilon^2)\,. \end{aligned}$$

Formal computations using the relation

$$\mathrm{div}_x\, \mathcal{H} \,+\, \nabla_x \Phi \cdot \partial_\theta \mathcal{H}_0 \,=\, 0$$

give rise to

$$(5.3) \qquad \left\{ \begin{aligned} & \partial_t \mathcal{H} + (\mathcal{H}\cdot\nabla_x)\mathcal{H} \,+\, \nabla_x \mathbf{p} \,+\, (\bar{\mathcal{H}}_0 \cdot \nabla_x \Phi)\, \partial_\theta \mathcal{H} \\ & \qquad - (\mathrm{div}_x\, \partial_\theta^{-1} \mathcal{H}^*)\, \partial_\theta \mathcal{H} \,+\, \partial_\theta P_0\, \nabla_x \Phi \,=\, 0\,, \qquad \mathrm{div}_x\, \bar{\mathcal{H}} \,=\, 0\,, \\ & \partial_t \Phi + (\bar{\mathcal{H}} \cdot \nabla_x)\Phi \,=\, 0\,, \qquad \nabla_x \Phi \cdot \mathcal{H}^* \,=\, 0\,. \end{aligned} \right.$$

Obviously, the weak limit of $\{\mathbf{u}^\varepsilon\}_\varepsilon$ is $\bar{\mathcal{H}}(t,x)$. Extract from (5.3) that

$$\partial_t \bar{\mathcal{H}} \,+\, (\bar{\mathcal{H}}\cdot\nabla_x)\bar{\mathcal{H}} \,+\, \langle \mathrm{div}_x\,(\mathcal{H}^* \otimes \mathcal{H}^*)\rangle \,+\, \nabla_x \mathbf{p} \,=\, 0\,, \qquad \mathrm{div}_x\, \bar{\mathcal{H}} = 0\,.$$

The system (5.3) can be completed with initial datas

$$(5.4) \qquad \mathcal{K}(x,\theta) \,:=\, \mathcal{H}(0,x,\theta)\,, \qquad \phi(x) \,:=\, \Phi(0,x)$$

which of course must satisfy the compatibility conditions

$$(5.5) \qquad \mathrm{div}_x\, \bar{\mathcal{K}} = 0\,, \qquad \nabla_x \phi \cdot \mathcal{K}^* = 0\,.$$

Now, it is always possible to further adjust $\mathcal{K}$ and ϕ so that

$$\text{(5.6)} \qquad \operatorname{curl} \langle \operatorname{div}_x (\mathcal{K}^* \otimes \mathcal{K}^*) \rangle \not\equiv 0 \, .$$

Then, the expression $\bar{\mathcal{H}}$ cannot be a solution of Euler equations. Therefore, we can see in the preceding manipulations a way to exhibit counter-examples to the concentration-cancellation property. However, this strategy clashes against two major difficulties :

1) Nothing guarantees that it is possible to find a particular solution $(\mathcal{H}, \Phi)$ of (5.3)-(5.4) satisfying (5.5)-(5.6). Of course, we can exhibit special solutions of (5.3). For instance, the expressions

$$\mathcal{H}_s(t,x,\theta) = {}^t\big(\mathcal{H}_s^{1*}(\theta), 0\big) \, , \qquad \Phi_s(t,x) = \gamma \, x_2 \, , \qquad \bar{\mathcal{H}}_0 \equiv 0 \, , \qquad P_0 \equiv 0$$

built with $\mathcal{H}_s^1 \in C^\infty(\mathbb{T};\mathbb{R})$ and $\gamma \in \mathbb{R}$ are convenient. But these choices do not give rise to (5.6). On the other hand, it seems illusive to solve the Cauchy problem (5.3)-(5.4) associated with general initial datas subjected to (5.5)-(5.6) even if these initial datas are small perturbations of $\mathcal{H}_s$ and Φ_s. Objections to do that are raised by D. Serre in [23]. The constraint (5.3) is non linear and the linearized equations along $\mathcal{H}_s$ and Φ_s are unstable in the sense of Hadamard.

2) Suppose that we dispose of a particular solution $(\mathcal{H}, \Phi)$ of (5.3)-(5.4) satisfying (5.5)-(5.6). Note first that the corresponding oscillating wave

$$\mathbf{u}_a^\varepsilon(t,x) \, = \, \mathcal{H}\big(t,x,\Phi(t,x)/\varepsilon\big) \, , \qquad \mathbf{p}_a^\varepsilon(t,x) \, = \, \mathbf{p}(t,x)$$

would not be an exact solution of (1.1), see Lemma 3.3. By the way, observe that the system (5.3) is a relaxed version of (1.16) since any solution of (1.16) yields a solution of (5.3) just by taking $\bar{\mathcal{H}}_0 \equiv 0$ and $P_0 \equiv 0$. Now, if $\mathcal{H}_0 \equiv 0$ and $P_0 \equiv 0$, we recover that $\operatorname{div}_x \mathcal{H}^* \equiv 0$ and that Φ is adjusted as in (1.13) which implies that $\operatorname{curl} \langle \operatorname{div}_x (\mathcal{K}^* \otimes \mathcal{K}^*) \rangle \equiv 0$. Thus, necessarily, any solution $(\mathcal{H}, \Phi)$ of (5.3)-(5.4) satisfying (5.5)-(5.6) furnishes only an approximate solution of (1.1). It remains to prove that it corresponds to an exact solution of (1.1). This means to absorb the small error terms which are induced. But such a work of justification is very delicate (and may not succed) because the underlying regime is supercritical so that strong instabilities are expected (in the spirit of the articles [8], [12], [14] and [19]).

Below, we do not face directly these two difficulties. On the one hand, we construct only very special solutions of (5.3). On the other hand, by restricting the discussion to the set $\mathcal{V}_0^\tau(\omega)$, we avoid many mechanisms of amplification (in fact the worst ones !).

Recall that $\{\breve{h}^\varepsilon\}_\varepsilon$ is a family which is well prepared for Euler equations. Consider the solution (on Ω^τ) of the Cauchy problem

$$(5.7) \qquad \begin{cases} \partial_t \breve{\mathbf{u}}^\varepsilon + (\breve{\mathbf{u}}^\varepsilon \cdot \nabla_x)\breve{\mathbf{u}}^\varepsilon = 0\,, \qquad \mathrm{div}_x\, \breve{\mathbf{u}}^\varepsilon = 0\,, \\ \breve{\mathbf{u}}^\varepsilon(0,x) = \breve{h}^\varepsilon(x)\,. \end{cases}$$

Here $\breve{h}^\varepsilon$ is given by an asymptotic expansion like (2.17) with $\breve{H}$ and $\breve{H}_0$ adjusted as in Lemmas 2.7 and 2.8. The analysis of chapter 4 shows that

$$\breve{\mathbf{u}}^\varepsilon(t,x) = \breve{U}^\varepsilon\big(t,x,\Phi(t,x)/\varepsilon\big)\,, \qquad \breve{U}^\varepsilon(t,x,\theta) = (\breve{U}+\varepsilon\,\breve{U}_1)(t,x,\theta)+O(\varepsilon^2)\,.$$

The expression $\breve{U}$ is identified through

$$\breve{\mathbf{U}}(t,x,\theta) = \breve{\mathbf{H}}\big(t,x,\theta+\mathbf{V}_0^1(t,x,\theta)\big)$$

where $\breve{\mathbf{H}}(t,x,\theta)$ is of the form (5.1) whereas $\mathbf{V}_0^1(t,x,\theta)$ is given by (4.14). By construction the profile $\breve{\mathbf{H}}$ is a solution of (1.16). Now, we can check that the expression $\breve{\mathbf{U}}$ is indeed a solution of (5.3) for some well adjusted functions $\bar{\mathcal{H}}_0$ and P_0. Moreover, the weak limit $\breve{\mathbf{u}}(t,x)$ of the family $\{\breve{\mathbf{u}}^\varepsilon\}_\varepsilon$ is $\breve{\mathbf{u}}(t,x) = \langle\breve{\mathbf{U}}\rangle(t,x)$. Since the divergence free relation is preserved when passing to the weak limit, the expression $\breve{\mathbf{u}}$ is subjected to

$$(5.8) \qquad \partial_t \breve{\mathbf{u}} + (\breve{\mathbf{u}}\cdot\nabla_x)\breve{\mathbf{u}} + \breve{\mathbf{f}} = 0\,, \qquad \mathrm{div}_x\, \breve{\mathbf{u}} = 0$$

for some function $\breve{\mathbf{f}}$. It remains to determine $\breve{\mathbf{f}}$. In general, as already noticed in Remark 3.2.2, we find $\breve{\mathbf{f}} \not\equiv 0$. In fact, we have even the following more precise information.

Lemma 5.1. *The construction can be adjusted so that curl* $\breve{\mathbf{f}} \not\equiv 0$.

<u>*Proof*</u> <u>*of*</u> <u>*Lemma*</u> <u>*5.1*</u>. Recall (4.16) which gives access to $\mathbb{U} = {}^t(\mathbb{U}^1,\mathbb{U}^2,\mathbb{U}^3)$ and in particular to $\breve{\mathbf{U}} \equiv {}^t(\mathbb{U}^1,\mathbb{U}^2)$. The function $\breve{\mathbf{U}}$ is such that

$$\partial_t\breve{\mathbf{U}} + (\breve{\mathbf{U}}\cdot\nabla_x)\breve{\mathbf{U}} + (\partial_2\Phi)^2\ \mathbb{U}^3\ \partial_\theta\breve{\mathbf{U}} = 0\,.$$

Extract the mean value to get

$$\breve{\mathbf{f}} = \langle(\breve{\mathbf{U}}^*\cdot\nabla_x)\breve{\mathbf{U}}^*\rangle + (\partial_2\Phi)^2\ \langle\mathbb{U}^3\ \partial_\theta\breve{\mathbf{U}}\rangle\,.$$

In view of (4.17), at time $t=0$, it remains

$$\breve{\mathbf{f}}_{|\,t=0} = \langle(\breve{H}^*\cdot\nabla_x)\breve{H}^*\rangle + \langle(\nabla_x\phi\cdot H_0)\ \partial_\theta\breve{H}\rangle$$

where $\breve{H} := \breve{\mathbf{H}}_{|\,t=0}$ is defined as in (2.33). In Lemma 2.7, select $f_0\equiv 0$ and take $\phi_0 \equiv 0$ so that

$$\breve{H}(x,\theta) = K\big(\phi(x),\theta\big)\ {}^t\big(1,-f\circ\phi(x)\big) + {}^t\big(0,g\circ\phi(x)\big)\,.$$

In particular, we have

$$\breve{H}^*(x,\theta) \,=\, K^*\big(\phi(x),\theta\big)\ {}^t\big(1,-f\circ\phi(x)\big)\,.$$

With (2.20), it follows immediately that

$$(\breve{H}^*\cdot\nabla_x)\breve{H}^* \,=\, -\,K^*(\phi,\theta)\,\partial_2\phi^{-1}\,(\nabla\phi^{\perp}\cdot\nabla_x)\big[K^*(\phi,\theta)\ {}^t(1,-f\circ\phi)\big] \,\equiv\, 0\,.$$

In Lemma 2.8, select $f_1\equiv 0$ and take $\phi_1\equiv 0$. We have to deal with

$$H_0 \,=\, \bar{H}_0(\phi) \,+\, K_0(\phi,\theta)\ {}^t(1,-f\circ\phi) \,+\, f'\circ\phi\ (\partial_\theta^{-1}K^*)(\phi,\theta)\ {}^t(0,1)\,.$$

Now, compute

$$\breve{\mathbf{f}}_{|\,t=0} \,=\, f'\circ\phi\ \langle K^*(\phi,\theta)^2\rangle\ \nabla\phi^{\perp}$$

which leads to

$$\operatorname{curl}\breve{\mathbf{f}}(0,x) \,=\, \Delta(l\circ\phi)(x)\,,\qquad l(z) \,:=\, \textstyle\int_0^z\, f'(y)\ \langle K^*(y,\theta)^2\rangle\ dy\,.$$

In general, this quantity is non trivial. For instance, choose

$$f(z) \,=\, -\,z\,,\qquad \phi(x) \,=\, x_2/(1+x_1)\,,\qquad K(z,\theta) \,=\, \sqrt{2}\ \cos(\pi\,\theta)\,.$$

In this special case, we find

$$\operatorname{curl}\breve{\mathbf{f}}(0,x) \,=\, -\,2\ x_2/(1+x_1)^3 \,\not\equiv\, 0\,. \qquad \square$$

Proof of Theorem 1.1. The family $\{\breve{\mathbf{u}}^\varepsilon\}_\varepsilon$ satisfies the conditions i) and ii) of Theorem 1.1. It converges weakly (as ε goes to zero) to $\mathbf{u}^0\equiv\breve{\mathbf{u}}\in C^1\big(\Omega^\tau\big)$. The function $\breve{\mathbf{u}}$ is subjected to (5.8) which involves the source term $\breve{\mathbf{f}}$. If $\breve{\mathbf{f}}$ can be expressed as the gradient of some scalar function $\mathbf{p}$, one has curl $\breve{\mathbf{f}}\equiv 0$. But, in view of Lemma 5.1, this condition is not always verified. Therefore, it can happen that $\mathbf{u}^0$ is not a solution to (1.1). □

Remark 5.2.1 - beyond Theorem 1.1. An improvement would be to replace in Theorem 1.1 the bounded domain Ω by $\mathbb{R}^2$ and the Sobolev space L^2_{loc} by L^p with any $p\in[1,+\infty]$. This would express that the operator solution associated with (1.1) is not closed for the weak L^p topology, replying to the question raised by L. Bertozzi and A. Majda [2]-p. 479.

Our approach does not give yet access to such a general result. Indeed, our key argument relies on the use of the nonlinear functional set $\mathcal{V}_0^\tau(\omega)$. At this level, two difficulties arise. On the one hand, the families $\{\breve{h}^\varepsilon\}_\varepsilon$ which are well prepared on $\mathbb{R}^2$ for (1.1) satisfy (2.36) which is not compatible with a global condition such as $\breve{h}^\varepsilon\in L^p(\mathbb{R}^2)$ except if $p=+\infty$. On the other hand, the hypothesis $f'\not\equiv 0$ is crucial in the proof of Lemma 5.1. However, when $f'\not\equiv 0$ or when $(f^\varepsilon)'\not\equiv 0$, there is no way to define on $\mathbb{R}^2$ a C^1 solution φ^ε to (2.41) (because of the formation of shocks).

Of course, one is tempted to *localize* the analysis. But, this forces into being faced with the problem of stability in the setting of (1.1), with some nonlinear pressure $\mathbf{p}$. Now, this is a much more delicate matter than what we did in Section 4. This is still an open question. $\diamond$

Remark 5.2.2 - interpretation related to the strong topology of L^p. Many mechanisms of amplifications concerning (1.1) and involving the L^p-norm have recently been exhibited [8], [12], [14], [19], see also Proposition 5.1 in [6]. The preceding method provides an alternative way to obtain such results. However, in so far as this subject has much been studied, this aspect will not be underlined here. $\diamond$

5.3 Small transversal perturbations

From now on, consider (1.2) with $\mathbf{f} \equiv 0$ or (1.1) with $\mathbf{p}$ constant.

Fix two integers J and m with $J \geq 2$ and $m \geq 1$. Choose any couple (H,ϕ) which is compatible with (1.2). Note $\mathbf{H}$ and Φ the expressions obtained through (3.1) and (3.2). Select a smooth phase $\zeta_0 \in C^\infty(\omega;\mathbb{R})$ satisfying

$$(5.9) \qquad \nabla_x \zeta_0 \in C_b^\infty(\omega;\mathbb{R}^2)\,, \qquad \nabla_x \phi^\perp \cdot \nabla_x \zeta_0 \not\equiv 0\,.$$

Take profiles $\hat{H}_j$ such that

$$\hat{H}_j(x,\theta,z) \in C_b^\infty(\omega \times \mathbb{T} \times \mathbb{T};\mathbb{R}^2)\,, \qquad \forall\, j \in \{0,\cdots,J-1\}\,.$$

We assume that the profile $\hat{H}_0$ is a non trivial well polarized function of z. More precisely, we impose

$$(5.10) \qquad \nabla_x \phi^\perp \cdot \partial_z \hat{H}_0 \not\equiv 0\,, \qquad \nabla_x \phi \cdot \partial_z \hat{H}_0 \equiv 0\,.$$

Clearly, there is no difficulty to construct such profiles $\hat{H}_0$. Now, consider a multidimensional oscillatory initial data of the form

$$(5.11) \qquad \begin{aligned} \hat{h}^\varepsilon(x) \,:=\, & H\big(x,\phi(x)/\varepsilon\big) \\ & + \textstyle\sum_{j=0}^{J-1} \varepsilon^{j+1}\, \hat{H}_j\big(x,\phi(x)/\varepsilon,\zeta_0(x)/\varepsilon\big) \,+\, \varepsilon^{J+1}\, \hat{r}h^\varepsilon(x) \end{aligned}$$

with a remainder $\hat{r}h^\varepsilon \in C_b^\infty(\omega)$ satisfying

$$(5.12) \qquad \sup \big\{ \parallel \hat{r}h^\varepsilon \parallel_{L^\infty(\omega)} + \varepsilon^{J-1} \ \parallel D_x\, \hat{r}h^\varepsilon \parallel_{L^\infty(\omega)}\,;\ \varepsilon \in]0,1] \big\} \,<\, \infty\,.$$

Easy computations using (5.10) indicate that

$$\mathrm{div}_x\, \hat{h}^\varepsilon(x) \,=\, O(1)\,, \qquad \det\big(D_x \hat{h}^\varepsilon(x)\big) \,=\, O(1)\,.$$

We consider now the equation (1.2) together with the initial condition

$$\hat{\mathbf{u}}^\varepsilon(0,x) \,=\, \hat{h}^\varepsilon(x)\,. \tag{5.13}$$

According to Lemma 2.3, the oscillating Cauchy problem (1.2)-(5.13) yields a family $\{\hat{\mathbf{u}}^\varepsilon\}_\varepsilon$ of C^1 solutions $\hat{\mathbf{u}}^\varepsilon$ to (1.2). The functions $\hat{\mathbf{u}}^\varepsilon$ are defined on some open domain of determinacy $\Omega^\tau \subset \mathbb{R}\times\mathbb{R}^2$ which does not depend on the parameter $\varepsilon \in]0,1]$. The structure of $\hat{\mathbf{u}}^\varepsilon$ encodes the interaction of the large amplitude wave $\mathbf{H}\big(t,x,\Phi(t,x)/\varepsilon\big)$ with small amplitude oscillations which can involve the phase ζ_0 transversal to ϕ. Here is what happens.

Proposition 5.1. *Assume (5.9), (5.10) and (5.10). The solution $\hat{\mathbf{u}}^\varepsilon$ to the Cauchy problem (1.2)-(5.13) inherits on Ω^τ the following asymptotic behavior (which is valid in L^∞)*

$$\begin{aligned} \hat{\mathbf{u}}^\varepsilon(t,x) \,&=\, \mathbf{H}\big(t,x,\Phi(t,x)/\varepsilon\big) \\ &\quad +\varepsilon\, \hat{\mathbf{H}}_0\Big(t,x,\Phi(t,x)/\varepsilon,\zeta\big(t,x,\Phi(t,x)/\varepsilon\big)/\varepsilon\Big) \,+\, O(\varepsilon^2) \end{aligned} \tag{5.14}$$

where the profiles

$$\hat{\mathbf{H}}_0(t,x,\theta,z)\in C^\infty_b(\Omega^\tau\times\mathbb{T}^2;\mathbb{R}^2)\,, \qquad \zeta(t,x,\theta)\in C^\infty_b(\Omega^\tau\times\mathbb{T};\mathbb{R})$$

contain non trivial oscillations in the sense that $\partial_z\hat{\mathbf{H}}_0\not\equiv 0$ and $\partial_\theta\zeta\not\equiv 0$. Moreover, the family of initial data $\{\hat{h}^\varepsilon\}_\varepsilon$ can be adjusted so that

$$\hat{h}^\varepsilon \,\in\, \mathcal{V}^\tau_0(\omega)\,, \qquad \forall\,\varepsilon\in]0,1]\,.$$

It follows that the transformation of (5.11) into (5.14) actually occurs for the incompressible Euler equations.

Proof of Proposition 5.1. Define the following functions of (x,θ,z)

$$\begin{aligned} \hat{V}^2_{j+1} \,&:=\, |\nabla_x\phi|^{-2}\;\nabla_x\phi^\perp\cdot\hat{H}_j\,, \qquad \forall\, j\in\{0,\cdots,J-1\}\,, \\ \hat{V}^3_j \,&:=\, |\nabla_x\phi|^{-2}\;\nabla_x\phi\cdot\hat{H}_j\,, \qquad \forall\, j\in\{0,\cdots,J-1\}\,. \end{aligned}$$

Consider also

$$r^{\varepsilon 2}(x) \,:=\, |\nabla_x\phi|^{-2}\;\nabla_x\phi^\perp\cdot\hat{r}h^\varepsilon\,, \qquad r^{\varepsilon 3}(x) \,:=\, |\nabla_x\phi|^{-2}\;\nabla_x\phi\cdot\hat{r}h^\varepsilon\,.$$

Introduce

$$\begin{aligned} \hat{V}^{\varepsilon 2}(x,\theta) \,&:=\, \textstyle\sum_{j=0}^{J-1}\,\varepsilon^{j+1}\;\hat{V}^2_{j+1}\big(x,\theta,\zeta_0(x)/\varepsilon\big)\,+\,\varepsilon^{J+1}\;\hat{r}^{\varepsilon 2}(x)\,, \\ \hat{V}^{\varepsilon 3}(x,\theta) \,&:=\, \textstyle\sum_{j=0}^{J-1}\,\varepsilon^{j}\;\hat{V}^3_{j}\big(x,\theta,\zeta_0(x)/\varepsilon\big)\,+\,\varepsilon^{J}\;\hat{r}^{\varepsilon 3}(x)\,. \end{aligned}$$

At time $t=0$, impose

$$\hat{\mathbf{V}}^\varepsilon(0,x,\theta) \,=\, \hat{V}^\varepsilon(x,\theta) \,:=\, {}^t\big(\,0\,,\,\hat{V}^{\varepsilon 2}(x,\theta)\,,\,\hat{V}^{\varepsilon 3}(x,\theta)\,\big)\,. \tag{5.15}$$

Observe that

$$\hat{\mathbf{u}}^\varepsilon(0,x) = \hat{h}^\varepsilon(x) = \hat{U}^\varepsilon\big(x,\phi(x)/\varepsilon\big)\,, \qquad \hat{U}^\varepsilon(x,\theta) := \Gamma\big(\varepsilon,0,x,\theta;\hat{V}^\varepsilon(x,\theta)\big)\,.$$

The definition of $\hat{V}^{\varepsilon 2}$ and the condition (5.10) imply that

$$\hat{V}^\varepsilon(x,\theta) \,=\, \hat{V}_0(x,\theta) \,+\, \textstyle\sum_{j=1}^{J-1} \varepsilon^j \; \hat{V}_j\big(x,\theta,\zeta_0(x)/\varepsilon\big) \,+\, \varepsilon^J \; r^\varepsilon(x) \tag{5.16}$$

where by convention $\hat{V}_0(x,\theta) := {}^t\big(0,0,\hat{V}_0^3(x,\theta)\big)$ and

$$\hat{V}_j(x,\theta,z) \,:=\, {}^t\big(0,\hat{V}_j^2(x,\theta,z),\hat{V}_j^3(x,\theta,z)\big)\,, \qquad \forall\, j\in\{1,\cdots,J-1\}\,.$$

Associate with the system (4.14) the initial data ${}^t(0,\hat{V}_0^3)$. This furnishes a solution ${}^t(\hat{\mathbf{V}}_0^1,\hat{\mathbf{V}}_0^3)$ of (4.14). Now solve on $\Omega^\tau\times\mathbb{T}$ the eiconal equation

$$\left\{\begin{array}{l} \partial_t\zeta \,+\, \big(\mathbf{H}(t,x,\theta+\hat{\mathbf{V}}_0^1)\cdot\nabla_x\big)\zeta \,+\, |\nabla_x\Phi|^2 \; \hat{\mathbf{V}}_0^3 \; \partial_\theta\zeta \,=\, 0\,, \\ \zeta(0,x,\theta) \,=\, \zeta_0(x)\,. \end{array}\right. \tag{5.17}$$

The description of the propagation of solutions $\hat{\mathbf{V}}^\varepsilon$ of (4.13) issued from oscillating initial data $\hat{V}^\varepsilon$ as in (5.16) is the matter of what is called multidimensional weakly nonlinear geometric optics [16]-[17]. Therefore, the stability Theorem of [13] can be applied. The solution $\hat{\mathbf{V}}^\varepsilon$ is such that

$$\hat{\mathbf{V}}^\varepsilon(t,x,\theta) \,=\, \hat{\mathbf{V}}_0(t,x,\theta) \,+\, \varepsilon\; \hat{\mathbf{V}}_1\big(t,x,\theta,\zeta(t,x,\theta)/\varepsilon\big) \,+\, O(\varepsilon^2)$$

where

$$\hat{\mathbf{V}}_0(t,x,\theta) \,=\, {}^t(\hat{\mathbf{V}}_0^1,0,\hat{\mathbf{V}}_0^3)(t,x,\theta)\,, \qquad \hat{\mathbf{V}}_1(0,x,\theta,z) \,=\, {}^t(0,\hat{V}_1^2,\hat{V}_1^3)(x,\theta,z)\,.$$

Recall that

$$\hat{\mathbf{u}}^\varepsilon(t,x) = \hat{\mathbf{U}}^\varepsilon\big(t,x,\Phi(t,x)/\varepsilon\big)\,, \qquad \hat{\mathbf{U}}^\varepsilon(t,x,\theta) \,:=\, \Gamma\big(\varepsilon,t,x,\theta;\hat{\mathbf{V}}^\varepsilon(t,x,\theta)\big)\,.$$

In particular

$$\hat{\mathbf{U}}^\varepsilon(t,x,\theta) \,=\, \mathbf{H}\big(t,x,\theta+\hat{\mathbf{V}}_0^1(t,x,\theta)\big) \,+\, \varepsilon\; \hat{\mathbf{H}}_0\big(t,x,\theta,\zeta(t,x,\theta)/\varepsilon\big) \,+\, O(\varepsilon^2)$$

$$\begin{aligned} \hat{\mathbf{H}}_0(t,x,\theta,z) \,:=\, & \partial_\theta\mathbf{H}\big(t,x,\theta+\hat{\mathbf{V}}_0^1(t,x,\theta)\big)\; \hat{\mathbf{V}}_1^1(t,x,\theta,z) \\ & +\hat{\mathbf{V}}_1^2(t,x,\theta,z)\; \nabla_x\Phi(t,x)^\perp \,+\, \hat{\mathbf{V}}_0^3(t,x,\theta)\; \nabla_x\Phi(t,x)\,. \end{aligned}$$

At time $t=0$, we find

$$\hat{\mathbf{H}}_0(0,x,\theta,z) \,=\, \hat{V}_1^2(x,\theta,z)\; \nabla_x\phi(x)^\perp \,+\, \hat{V}_0^3(x,\theta)\; \nabla_x\phi(x)\,.$$

Because of (5.10), we are sure that $\partial_z\hat{V}_1^2 \not\equiv 0$. Therefore, by continuity, for t small enough, we still have $\partial_z\hat{\mathbf{H}}_0 \not\equiv 0$. Now, exploit again (5.9) to obtain

$$\partial_t\,(\partial_\theta\zeta)(0,\cdot) \,=\, -\,\partial_\theta s^*\; (\nabla_x\phi^\perp\cdot\nabla_x)\zeta_0 \,\not\equiv\, 0\,.$$

It follows that $\partial_\theta \zeta \not\equiv 0$. The first part of Proposition 5.1 is shown.

Examples of families $\{\breve{h}^\varepsilon\}_\varepsilon$ which are well prepared on ω for incompressible Euler equations are given in Lemma 2.7. Follow the same procedure as in the proof of Lemma 2.7 except at the level of (2.40). There, seek solutions φ^ε to (2.39) having the form

$$\varphi^\varepsilon(x) \,=\, \phi(x) \,+\, \varepsilon \;\phi^\varepsilon\big(x, \phi(x)/\varepsilon, \zeta_0(x)/\varepsilon\big)$$

where ϕ^ε can be expanded according to

$$\phi^\varepsilon(x,\theta,z) \,=\, \textstyle\sum_{j=0}^{J+1} \varepsilon^j \;\phi_j(x,\theta,z) \,+\, \varepsilon^{J+2} \; r\phi^\varepsilon(x,\theta,z)\,.$$

Impose $\partial_z \phi_0 \equiv 0$ and $\partial_z \phi_1 \not\equiv 0$ so that (Lemma 2.8)

$$\partial_z \hat{H}_0(x,\theta,z) \,=\, -\,\partial_2 \phi^{-1} \;\partial_z \phi_1(x,\theta,z)\; \partial_\theta K \;\nabla_x \phi^\perp$$

Obviously, the restrictions mentioned in (5.10) are verified. This observation completes the proof. □

Remark 5.3.1 - improvement of Proposition 5.1. The preceding analysis allows to take into account, already at time $t=0$, an oscillating phase $\zeta_0\big(x,\phi(x)/\varepsilon\big)$. The presentation has been here intentionally simplified. It is to underline the creation of a new frequency (by interaction of waves having the same frequencies). As explained in the introduction, this phenomenon has no link with resonances. ◇

Remark 5.3.2 - transfer of energy. Examine how the square $\mathcal{F}(\hat{\mathbf{u}}^\varepsilon)^2$ of the Fourier transform $\hat{\mathbf{u}}^\varepsilon$ of $\mathbf{u}^\varepsilon$ is distributed. At time $t=0$, it is divided amongst the two characteristic wave numbers $k \simeq 1$ and $k \simeq \varepsilon^{-1}$. In view of (5.14), this situation does not persist. At a time $t>0$, the concentration is around the three wave numbers $k \simeq 1$, $k \simeq \varepsilon^{-1}$ and $k \simeq \varepsilon^{-2}$. This spontaneous apparition of a new frequency can be interpreted as a basic mechanism of turbulent phenomena. ◇

Remark 5.3.3 - other frequencies. The phase ζ can as well oscillate with a frequency given by $\eta(\varepsilon)$ where $\eta:]0,1] \longrightarrow]0,1]$ is a decreasing function of ε which tends to 0 as ε goes to 0, such as $\eta(\varepsilon) = \varepsilon^j$ with $j \in \mathbb{N}_*$. Then, the frequencies of the interacting waves are multiplied. Thus, the resulting wave oscillates with the frequency $\big(\varepsilon\,\eta(\varepsilon)\big)^{-1}$. ◇

Remark 5.3.4 - other interpretation of what happens. Suppose that the phases ϕ and ζ_0 are linear. For instance, select

$$\phi(x) \,=\, x_2\,, \qquad \zeta_0(x) \,=\, x_1\,.$$

Consider (1.2) with $\mathbf{f} \equiv 0$. Assume also that all the profiles and the remainder which appear in (5.11) are periodic with period $\sqrt{\varepsilon}$ and $1/\sqrt{\varepsilon}$ respectively with respect to the *slow* variables x_1 and x_2. Make the following transformations

$$x_2 \mid \sqrt{\varepsilon}\ \tilde{x}_2\,, \qquad x_1 \mid \tilde{x}_1/\sqrt{\varepsilon}\,, \qquad \mathbf{u}^1 \mid \tilde{\mathbf{u}}^1/\sqrt{\varepsilon}\,, \qquad \mathbf{u}^2 \mid \sqrt{\varepsilon}\ \tilde{\mathbf{u}}^2\,.$$

Observe that (1.2) is conserved

$$\partial_t \tilde{\mathbf{u}}^\varepsilon \,+\, (\tilde{\mathbf{u}}^\varepsilon \cdot \nabla_x)\tilde{\mathbf{u}}^\varepsilon \,=\, 0 \tag{5.18}$$

whereas the initial data are transformed according to

$$\tilde{\mathbf{u}}^\varepsilon(0,x) \,=\, \sqrt{\varepsilon}\ \tilde{H}^0(\tilde{x},\tilde{x}_2/\sqrt{\varepsilon}) \,+\, \varepsilon^{3/2}\ \tilde{H}^1(\tilde{x},\tilde{x}_2/\sqrt{\varepsilon},\tilde{x}_1/\varepsilon^{3/2}) \,+\, O(\varepsilon^{5/2})\,.$$

Introduce the fast variable $\tilde{y}_2 := \tilde{x}_2/\sqrt{\varepsilon}$ and the new unknown $\tilde{\mathbf{v}}^\varepsilon := \tilde{\mathbf{u}}^\varepsilon/\sqrt{\varepsilon}$ so that we have to deal with an expression like

$$\tilde{\mathbf{v}}^\varepsilon(0,x,\tilde{y}_2) \,=\, \tilde{H}^0(\tilde{x},\tilde{y}_2) \,+\, \varepsilon^{3/2}\ \tilde{H}^1(\tilde{x},\tilde{y}_2,\tilde{x}_1/\varepsilon^{3/2}) \,+\, O(\varepsilon^{5/2})$$

and with the equation

$$\partial_t \tilde{\mathbf{v}}^\varepsilon \,+\, \sqrt{\varepsilon}\ (\tilde{\mathbf{v}}^\varepsilon \cdot \nabla_{\tilde{x}})\tilde{\mathbf{v}}^\varepsilon \,+\, \tilde{\mathbf{v}}^\varepsilon\ \partial_{\tilde{y}_2}\tilde{\mathbf{v}}^\varepsilon \,=\, 0\,. \tag{5.19}$$

In this interpretation, $\tilde{H}^0(\tilde{x},\tilde{y}_2)$ appears as a basic state which is modified by a $O(\varepsilon^{3/2})$ perturbation oscillating with a $O(\varepsilon^{-3/2})$ frequency. This is exactly the framework of weakly nonlinear geometric optics. And this is the reason why we said in the introduction that, all things considered, the matter of this paper is technically equivalent to deal with a problem of *superposition* of weakly nonlinear geometric optics.

This point of view explains very well why a new frequency appears (this is simply the slow variable $\tilde{y}_2$ in the eiconal equation related to (5.19)). But, of course, this approach must be adopted with many precautions since, in the procedure, the domains of definition of the functions are changed ! $\diamond$

Remark 5.3.5 - interaction of large amplitude transversal oscillating waves. The problem to find solutions of (2.39) having the form (2.40) can be solved (on a domain which does not depend on $\varepsilon \in]0,1]$) by seeking some profile $\tilde{\phi}^\varepsilon(x,\theta,z) \in C^\infty(\omega \times \mathbb{T}^2;\mathbb{R})$ smooth in $\varepsilon \in [0,1]$ and such that

$$\phi^\varepsilon\big(x,\phi(x)/\varepsilon\big) \,=\, \tilde{\phi}^\varepsilon\big(x,\phi(x)/\varepsilon,\zeta(x)/\varepsilon\big)\,, \qquad \partial_z\tilde{\phi}^0(x,\theta,z) \not\equiv 0\,.$$

This corresponds to the construction of initial data $\breve{h}^\varepsilon$ which have the form

$$\breve{h}^\varepsilon(x) \,=\, \breve{H}^\varepsilon\big(x,\phi(x)/\varepsilon,\zeta(x)/\varepsilon\big)\,, \qquad \breve{H}^\varepsilon \in C^\infty([0,1]\times\omega\times\mathbb{T}^2;\mathbb{R}^2)$$

and which are well prepared for (1.1). This clearly indicates that the method developed here can be pushed forward to incorporate the interaction of some *very special* large amplitude multi-phase oscillations. To this end, a possible approach could be to repeat the present analysis at the level of (4.13) which again is a Burger type equation. The new difficulties is the presence of variable coefficients and of more space variables. We are satisfied here to touch on this delicate subject through the present allusion. $\diamond$

Remark 5.3.6 - other formal solutions. The data ϕ and H have been adjusted in a very special way. It would be interesting to extend the formal WKB calculus in order to incorporate more general phases ϕ and profiles H. This can reveal many other complex phenomena. For sure, the picture must be completed with the cascade of phases observed in [6], and no only. $\diamond$

Remark 5.3.7 - prospects. The stability analysis of this paper relies on a last resource. Indeed, concerning (1.1), it is restricted to the set $\mathcal{V}_0^\tau(\omega)$. The question to know what happens when the $O(\varepsilon)$ perturbations push the initial oscillating data out of $\mathcal{V}_0^\tau(\omega)$, which means to deal with the pressure term, is at the moment open. $\diamond$

References

[1] Alinhac, S. *Temps de vie et comportement explosif des solutions d'équations d'ondes quasi-linéaires en dimension deux. II*, Duke Math. J. 73 (1994), no. 3, 543–560.

[2] Bertozzi, Andrea L. ; Majda, Andrew J. *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, 27, Cambridge University Press (2002).

[3] Biryuk, A. *On Multidimensional Burgers Type Equations with Small Viscosity* , Contributions to current challenges in mathematical fluid mechanics, 1–30, Adv. Math. Fluid Mech., Birkhäuser, Basel (2004).

[4] Chemin, J.-Y. *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, 14, Oxford University Press, New York, 187 pp. (1998).

[5] Cheverry, C. *Propagation of Oscillations in Real Vanishing Viscosity Limit*, Commun. Math. Phys. 247, 655-695 (2004).

[6] Cheverry, C. *Cascade of phases in turbulent flows*, to appear in Bulletin de la SMF, 52 pp.

[7] Cheverry, C. *Sur la propagation de quasi-singularités*, exposé à l'école Polytechnique (2004).

[8] C. Cheverry, O. Guès, G. Métivier, *Oscillations fortes sur un champ linéairement dégénéré*, Ann. Scient. Ec. Norm. Sup., 4^{e} série, t. 36, 2003, p. 691 à 745.

[9] Cheverry, C. ; Guès, O. ; Métivier, G. *Large amplitude high frequency waves for quasilinear hyperbolic systems*, Adv. Differential Equations 9, no. 7-8, 829–890 (2004).

[10] Delort, J.-M. *Existence de nappes de tourbillon en dimension deux*, J. Am. Math. Soc. 4, 553–586 (1991).

[11] DiPerna, R.-J. ; Majda, Andrew J. *Oscillations and Concentrations in Weak Solutions of the Incompressible Fluid Equations*, Commun. Math. Phys. 108, 667–689 (1987).

[12] S. Friedlander, W. Strauss, M. Vishik, *Nonlinear instability in an ideal fluid*, Ann. Inst. Henri Poincaré, Vol.14, no. 2, (1997), 187-209.

[13] O. Guès, *Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires.* Asymp. Anal.,6, 241-269, 1993.

[14] E, Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Comm. Pure Appl. Math. 53 (2000), no. 9, 1067–1091.

[15] Gallagher, I. ; Saint-Raymond, L. *Asymptotic results for pressureless magneto–hydrodynamics*, arxiv, math.AP/0312021.

[16] Joly, J.-L. ; Métivier, G. ; Rauch, J. *Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves*, Duke Math. J. 70, no. 2, 373–404 (1993).

[17] Joly, J.-L. ; Métivier, G. ; Rauch, J. *Coherent and focusing multidimensional nonlinear geometric optics*, Ann. Sci. École Norm. Sup. (4) 28, no. 1, 51–113 (1995).

[18] Joly, J.-L. ; Métivier, G. ; Rauch, J. *Nonlinear oscillations beyond caustics*, Comm. Pure Appl. Math., 49, no. 5, 443–527 (1996).

[19] Lebeau, G. *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège 70 (2001), no. 4-6, 267–306 (2002).

[20] Majda, Andrew J. *Compressible fluid flow and systems of quasilinear equations in several space variables*, Applied Mathematical Sciences, vol 53, Springer-Verlag, New York (1984).

[21] Poupaud, F. *Global smooth solutions of some quasi-linear hyperbolic systems with large data.* Ann. Fac. Sci. Toulouse Math. (6) 8, no. 4, 649–659 (1999).

[22] Schochet, S. *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Comm. Partial Differential Equations 20, 1077-1104 (1995).

[23] Serre, D. *Oscillations non-linéaires hyperboliques de grande amplitude*, Nonlinear variational problems and partial differential equations (Isola d'Elba, 1990), 245–294, Pitman Res. Notes Math. Ser., 320, Longman Sci. Tech., Harlow, 1995.

[24] Yudovich, V.I. *Non-stationary flow of an incompressible liquid*, Zh. Vychisl. Mat. Fiz. 3, 1032-1066 (1963).